CHARTA LEARNING · LESSON NOTES

The one-period binomial model: replication and risk-neutral valuation

CFA Level I · Derivatives

You will learn: Price a call or a put from a one-period, two-outcome tree in two ways, by replication with shares and a loan and by risk-neutral weights, and use the price to spot and trade a mispriced option.

Running example: Examples follow Juniper Tools, a made-up company with a share at 50 today. After one period it is 60 (up 20%) or 42.50 (down 15%). The call has strike 52, so it pays 8 if up and 0 if down. The risk-free rate is 3% for the period.

1. The one-period tree and its rules

- Binomial means two outcomes: after one period the share is at an up price or a down price. It is unrealistic but a building block; many small steps give a realistic model.
- Rules: one period; a European option (used only at expiry); no dividends; borrowing and lending at the same risk-free rate; no trading costs, short selling allowed; and no arbitrage, meaning no sure profit with no risk and no own money.
- A call pays max(share price − strike, 0), a put max(strike − share price, 0). The option's value today sits between its two possible payoffs.

JUNIPER TOOLS: SHARE 50, UP FACTOR 1.20, DOWN FACTOR 0.85, CALL STRIKE 52		
Share in the up state	50 × 1.20	**60.0**
Share in the down state	50 × 0.85	**42.5**
Call payoff, up	60 − 52	**8**
Call payoff, down	max(42.50 − 52, 0)	**0**

2. Replication: the hedge ratio and the loan

- Replication copies the option with shares and a loan, chosen to pay what the option pays in both states. By the law of one price, identical payoffs must cost the same, so the copy's cost today is the option's no-arbitrage value.
- The hedge ratio h is the number of shares held per option. It is not a probability and does not depend on how likely the up move is. For a call it lies between zero and one; for a put it is negative (sell shares short and lend).
- The shares fix the gap between the two payoffs; the loan fixes the level. Borrow the present value of what the shares are worth in the down state less the option's down payoff. The option's value is the cost of the shares minus the loan.

h = (c_up − c_down) ÷ (S_up − S_down); call = h × S − loan; loan = (h × S_down − c_down) ÷ (1 + r)

REPLICATING JUNIPER'S CALL		
Hedge ratio	(8 − 0) ÷ (60 − 42.50) = 8 ÷ 17.5	**0.4571**
Shares in the down state; loan today	0.4571 × 42.50 = 19.429; ÷ 1.03	**19.429; 18.863**
Up state: shares less repayment	27.429 − 19.429	**8, matches the call**
Cost of the shares	0.4571 × 50	**22.857**
Fair call price	22.857 − 18.863	**3.994**

Watch out: Forgetting the loan; forgetting to discount the loan; wrong sign of the hedge ratio; mixing up the up and down states.

3. A put, and put-call parity as a check

- A put is built the same way. It pays 0 if up and 9.5 if down (strike 52), so the hedge ratio is negative: the copy sells shares short and lends the proceeds. Always check a copy in both states.
- Put-call parity is a bonus check: call minus put equals the share price minus the present value of the strike.

JUNIPER'S PUT, STRIKE 52		
Hedge ratio	(0 − 9.5) ÷ 17.5	**−0.5429**
Cash from shorting shares	0.5429 × 50	**27.143**
Cash lent	(23.071 + 9.5) ÷ 1.03	**31.623**
Fair put price	31.623 − 27.143	**4.480**
Up state / down state: deposit − buy back	32.571 − 32.571 / 32.571 − 23.071	**0 / 9.5**
Parity: call − put; share − PV of strike	3.994 − 4.480; 50 − 52 ÷ 1.03	**−0.485 both**

4. The odds do not matter; trading a mispriced option

- The price of the copy uses only the tree and the interest rate. Today's share price already reflects the market's view of the odds, so a different view about the odds is really a view about the share.
- If the option is too dear, sell it and buy the copy. If too cheap, buy it and sell the copy. At expiry everything cancels in both states, so the cash collected today is a sure profit. Traders pile in until the price returns to fair value.

CALL FAIR AT 3.994		
Traders with 30% / 80% chance of a rise: expected payoff	0.3 × 8 / 0.8 × 8	**2.4 / 6.4, but copy cost 3.99 for both**
Market price 4.50: sell call, buy 0.4571 shares, borrow	4.500 − 22.857 + 18.863	**+0.506 today**
At expiry, up: shares less loan and call	27.429 − 19.429 − 8	**0**
At expiry, down: shares less loan	19.429 − 19.429	**0**
Market price 3.50: buy call, short shares, lend	−3.50 + 22.857 − 18.863	**+0.494 today**

Watch out: Using your own forecast of the chance of a rise.

5. Risk-neutral valuation

- The second route uses risk-neutral probabilities: weights, not forecasts, chosen so the share is expected to earn exactly the risk-free rate. Then the option's expected payoff can be discounted at the risk-free rate, so no unknown risk-adjusted rate is needed.
- The up weight π uses only the up factor u, the down factor d and the rate. The down weight is 1 − π. The risk-free growth must sit between d and u, or one asset always beats the other, which is an arbitrage.
- Risk neutral does not mean investors ignore risk. Their risk aversion is already in the share price. The weights are a computational device and give the same answer as replication, since both rest on no arbitrage.

> **π = (1 + r − d) ÷ (u − d); option = [π × payoff_up + (1 − π) × payoff_down] ÷ (1 + r)**

JUNIPER, U = 1.20, D = 0.85, R = 3%		
Up weight π	(1.03 − 0.85) ÷ (1.20 − 0.85) = 0.18 ÷ 0.35	**0.5143**
Down weight	1 − 0.5143	**0.4857**
Expected share price; growth	0.5143 × 60 + 0.4857 × 42.50 = 51.5; 51.5 ÷ 50	**1.03**
Call	0.5143 × 8 ÷ 1.03	**3.994**
Put	0.4857 × 9.5 ÷ 1.03	**4.480**

6. Both routes on a fresh share

- The two routes always agree, and the real chance of a rise never appears. With more periods, repeat the one-step valuation working backwards from expiry.
- The naive answer (real chance × payoff, discounted) is right only where the real chance equals the risk-neutral weight, 51.4% here. At every other chance it is wrong.

SHARE 100 TO 125 OR 80, STRIKE 105, R = 4%

Hedge ratio	20 ÷ 45	**0.4444**
Replication: shares less loan	44.444 − 34.188	**10.256**
Risk-neutral weight π	(1.04 − 0.80) ÷ 0.45	**0.5333**
Risk-neutral value	0.5333 × 20 ÷ 1.04	**10.256**

Key takeaways

- A one-period tree gives the share two prices; the option's payoff in each state follows.
- Replication: hold h = (c_up − c_down) ÷ (S_up − S_down) shares and borrow the rest; the copy's cost is the option's price.
- Risk-neutral route: π = (1 + r − d) ÷ (u − d), then discount the weighted payoff at the risk-free rate. Both routes agree.
- The real chance of a rise never enters the price; it is already in today's share price.
- A mispriced option is an arbitrage: sell what is dear, buy what is cheap, hedge with the copy.
- Risk neutral is a device, not an assumption that investors ignore risk.

Quick check

1. Share 40 → 48 or 34, r = 2%, call strike 40. Hedge ratio and call value?
   Answer: 8 ÷ 14 = 0.5714. Call = 3.81.
2. The real chance of a rise goes from 40% to 60%. What happens to the call value?
   Answer: Nothing. The real odds never enter the price.
3. Same share, put struck at 40. Put value?
   Answer: Down weight 0.5143 × 6 ÷ 1.02 = 3.03.