CHARTA LEARNING · LESSON NOTES

Pricing forwards and futures: arbitrage and the cost of carry

CFA Level I · Derivatives

You will learn: Price a forward from the cost of carrying the asset rather than a forecast, value a forward after it is signed, find the rate on a forward rate agreement, and explain why futures and forward prices can differ.

Running example: Examples follow Kestrel Capital, a made-up trading desk. Every price and rate is a round number made up for teaching, with discrete (not continuous) compounding.

1. No arbitrage and the simplest forward price

- A forward contract is an agreement to buy or sell the underlying at a fixed forward price on a fixed date. The price for immediate delivery is the spot price. The long agrees to buy, the short to sell, and nothing is paid at the start.
- An arbitrage is a certain profit with no risk and no own money. Real markets close such gaps fast, so we assume none. That fixes the forward price: it must equal the cost of replicating the forward by borrowing, buying the asset and holding it.
- With no income and no storage cost the forward price is the spot price compounded at the risk-free rate. It uses no forecast of the future spot price, only the cost of carrying. A longer contract has a higher forward price.

> **$F_0 = S_0 \times (1 + r)^T$**

SPOT 100, RISK-FREE RATE 5%, NO INCOME, NO STORAGE

One-year forward	100×1.05	**105**
Two-year forward	100×1.05^2	**110.25**
Quoted at 108: sell forward, borrow 100, buy asset; deliver and repay	108 − 105	**$3 riskless (cash and carry)**
Quoted at 102: buy forward, short asset, lend 100	105 − 102	**$3 riskless (reverse cash and carry)**

Watch out: Treating the forward price as a prediction of the future spot price.

2. Costs and benefits of carry

- Together the costs and benefits of holding are the cost of carry. Costs raise the forward price: interest, and storage and insurance for physical goods. Benefits lower it: dividends and coupons (paid to today's holder, not the forward buyer) and convenience yield (having the physical thing on hand).
- Take the present value of each: a payment divided by (1 + r) raised to the years until it is made. Subtract the benefits, add the costs, then compound at the risk-free rate.
- If costs of carry exceed benefits, the forward is above spot (contango). If benefits exceed financing and storage costs, the forward is below spot (backwardation).

$F_0 = (S_0 - \text{PV benefits} + \text{PV costs}) \times (1 + r)^T$

SHARE, GOLD AND OIL (RATE 5%, ONE YEAR)

Share, spot 100, dividend 2 at delivery: PV of dividend	2 ÷ 1.05	**1.90**
Forward	(100 − 1.90) × 1.05	**103.00 (check 105 − 2)**
Gold, spot 2,000, storage 20 at year end: PV; forward	20 ÷ 1.05 = 19.05; (2,000 + 19.05) × 1.05	**2,120**
Gold quoted at 2,090: buy forward, sell gold held, lend	2,120 − 2,090	**$30 riskless**
Oil, spot 80, PV storage 1, PV convenience yield 6	(80 − 6 + 1) × 1.05	**78.75, below spot: backwardation**

Watch out: Adding benefits and subtracting costs the wrong way round.

3. The value of a forward after it starts

- The price is the number in the contract and never changes. The value is what the contract is worth today and moves all the time. At the start the value is zero for both sides.
- Step one: find today's forward price for the remaining time, F_t. Step two: compare it with the price signed, F_0; if higher, the long gains the difference. Step three: that gap is paid at delivery, so discount it to today. The short's value is the same with opposite sign.
- Compare with the current forward price for the same delivery date, not with spot. At expiry no time is left, so the value is the spot price then minus the signed price.

V_t (long) $= [F_t - F_0] \div (1 + r)^{(T - t)}$

SIGNED AT 104 (SPOT 100, RATE 4%); SIX MONTHS LATER SPOT IS 108

Forward for time left	108 × 1.04^0.5	**110.14**
Gap to signed price	110.14 − 104	**6.14**
Discount half a year: value to the long	6.14 ÷ 1.04^0.5	**6.02**
Value to the short		**−6.02**

Watch out: Mixing up price and value; forgetting the last step, discounting the gain back to today.

4. Forward rate agreements and implied forward rates

- A forward rate agreement (FRA) fixes a rate today for borrowing that starts in the future. It settles in cash and no loan is made under it. It is used as insurance on a future interest rate.
- No arbitrage: lending for the long period must give the same money as lending for the short period, then for the gap at a rate agreed today. That is the implied forward rate. The FRA rate equals it, so the FRA is worth zero when signed.
- Money market rates use a 360-day year. The long (pays fixed, receives floating) gains when rates rise and suits a future borrower; the short gains when rates fall and suits a future lender. It settles at the start of the period, so the payment is discounted.

> **Forward rate = [(1 + R_b × b/360) ÷ (1 + R_a × a/360) − 1] × 360 ÷ (b − a)**

REDWOOD FOODS: 3 × 6 FRA (90 DAYS 4.0%, 180 DAYS 4.5%), NOTIONAL $10M

Growth over 180 days / 90 days	1 + 0.045 × 180/360; 1 + 0.04 × 90/360	**1.0225 / 1.0100**
Rate for the gap, annualised	1.0225 ÷ 1.01 − 1 = 1.2376%; × 360/90	**4.95%**
Day 90 rate is 5.5%: owed at day 180	(5.50% − 4.9505%) × 90/360 × 10m	**$13,738**
Paid to the long at day 90	13,738 ÷ (1 + 0.055 × 90/360) = 13,738 ÷ 1.01375	**$13,551**
Annual rates: 1-year 3%, 2-year 4%	$1.04^2 \div 1.03 - 1 = 1.0816 \div 1.03 - 1$	**5.01%**

5. Futures against forwards

- A futures contract is a standard forward traded on an exchange with a clearing house and margin, and marked to market daily. A forward is private, tailored, settled once at the end, and carries the other side's credit risk.
- Both are priced from the same cost of carry. Futures gains arrive in cash daily and can be invested, and losses must be funded. If interest rates are constant these effects cancel and the prices are equal.
- Prices differ only when rates move with the underlying's price. Positive correlation: futures price above forward price. Negative correlation (as with bonds): futures price lower. No correlation: equal. Taxes, liquidity and credit risk also differ. For the exam, normally use the forward formula for both.

WHY A LONG FUTURE CAN BE WORTH MORE (±$10,000 MOVE, 6 MONTHS; RATES 6% IF PRICE UP, 2% IF DOWN)

Price up: gain invested	10,000 × 6% × ½	**+$300**
Price down: loss funded	10,000 × 2% × ½	**−$100**
Average edge for futures	(300 − 100) ÷ 2	**+$100**

Key takeaways

- The forward price comes from the cost of replicating it, $F_0 = S_0(1 + r)^T$, not from a forecast.
- Costs of carry (interest, storage, insurance) raise the price; benefits (dividends, coupons, convenience yield) lower it.
- Price is fixed; value starts at zero and equals the present value of $F_t - F_0$.
- A forward rate agreement locks in the implied forward rate and settles in cash, discounted.
- Futures and forwards differ in price only when interest rates move with the underlying.

Quick check

1. Spot 50, rate 4%, dividend 1 at delivery in one year. Forward?
   Answer: 50 × 1.04 − 1 = $51.
2. Signed at 100. Forward now 106, discount factor 0.99. Value to long?
   Answer: (106 − 100) × 0.99 = 5.94.
3. Rates and price negatively correlated. Futures vs forward?
   Answer: Futures price is lower.