CHARTA LEARNING · LESSON NOTES

Convexity, portfolio duration and curve-based risk measures

CFA Level I · Fixed Income

You will learn: Estimate price changes with duration and convexity, measure portfolio duration, choose between modified, effective, key rate and empirical duration, and use the duration gap against an investment horizon.

Running example: Worked examples follow one made-up investor, the Alder Pension Fund. Its main bond is a 10-year bond with a 4% coupon paid twice a year (30/360 day count), price 100 per 100 of par at a 4% yield. Figures are examples.

1. Convexity: the bend that duration misses

- Price moves opposite to yield. Duration is the slope, a straight-line estimate; the true relationship is a curve. Convexity measures the bend and matters most when yields move a long way.
- The curve sits above the duration line on both sides, so a fall in yield gains more than a rise loses. For an option-free bond, convexity is always positive and the convexity term is added whichever way yields move.
- Do not drop the one half or the square. Doubling the move makes the correction four times as big. Of two bonds with the same duration, the one with more convexity gains more and loses less, so investors accept a lower yield for it.

Convexity ≈ (PV− + PV+ − 2 × PV0) ÷ (PV0 × Δy²); %ΔPrice ≈ −MD × Δy + ½ × C × (Δy)²

ALDER BOND, YIELD 4% ± 1%; PRICES 108.5843 AND 92.2054

Sum of the two prices	108.5843 + 92.2054	**200.7897**
Less twice today's price	200.7897 − 200	**0.7897**
Convexity	0.7897 ÷ 0.01	**78.97**
Duration from the same prices	16.3789 ÷ 2.00	**8.19**
Yield falls 4% to 2%: estimate	8.19 × 2.00% + ½ × 78.97 × 0.02² = 16.38 + 1.58	**17.96% (true 18.05%)**
Yield rises 2 points: with convexity	−16.38 + 1.58	**−14.80% (true −14.88%)**

Watch out: Subtracting convexity when yields rise; dropping the ½ or the square.

2. Portfolio duration

- Portfolio duration is the market-value-weighted average of the bond durations. Use market value, not face value, because market value is what is at risk today. It gives the approximate percentage change in the portfolio for a one percentage point parallel move.
- It is right only for a parallel shift, where every yield moves equally. Real curves steepen, flatten and twist, so two portfolios with the same duration can behave very differently.

Portfolio duration = w1 × D1 + w2 × D2 + … + wn × Dn

ALDER HOLDS 100M IN THREE BONDS

Short bond: 50m	0.50 × 3.0	**1.5**
Medium bond: 30m	0.30 × 6.0	**1.8**
Long bond: 20m	0.20 × 11.0	**2.2**
Portfolio duration; loss if all yields rise 1%	1.5 + 1.8 + 2.2	**5.5; 5.5m**
Bullet (duration 7.0) loss, 7-year yield +0.25%	7.0 × 0.25%	**1.75%**
Barbell (duration 7.0) loss, 2y +0.00, 12y +0.50	½ × 12.0 × 0.50%	**3.00%**

Watch out: Same duration does not mean same risk.

3. Effective duration for bonds with options

- Modified duration assumes fixed cash flows. A callable bond, a putable bond and a mortgage-backed security have cash flows that change with rates (the issuer calls, the investor puts, homeowners prepay).
- Effective duration shifts the whole benchmark curve up and down, lets a pricing model reprice the bond (so calls and prepayments are reflected), and divides by twice today's price times the shift. Use it for callable, putable and mortgage bonds; use modified duration only for option-free bonds.
- A callable bond has negative convexity: as yields fall, its price flattens because the issuer will call. A mortgage-backed security behaves the same way, and its effective duration can even turn negative.

Effective duration = (PV− − PV+) ÷ (2 × PV0 × Δcurve)

CALLABLE BOND, MODEL PRICES, CURVE SHIFTED 1% EACH WAY

Price if curve falls 1%		**102.20**
Price if curve rises 1%		**92.60**
Price today		**97.80**
Effective duration	9.60 ÷ (2 × 97.80 × 0.01)	**4.91**
Twin with no call option	modified duration	**8.19**

Watch out: Using modified duration for a bond with an embedded option; it overstates sensitivity.

4. Key rate duration

- A parallel shift is not all that happens: the curve also twists. Key rate duration asks how much the value changes if only the yield at one maturity moves and every other stays fixed, then repeats this for 2, 5, 10 and 30 years.
- The key rate durations of a portfolio add up to approximately its effective duration. The individual figures show where on the curve the risk sits, so a steepening or flattening can be hedged directly.

KRD at one maturity = (PV− − PV+) ÷ (2 × PV0 × Δy)

ALDER PORTFOLIO, PER 100 OF VALUE

Value if 10-year yield falls / rises 1%		**103.45 / 96.65**
Key rate duration, 10 years	6.80 ÷ (2 × 100 × 0.01)	**3.40**
KRDs at 2y; 5y; 10y; 30y	sum = 5.5, the portfolio duration	**0.5; 1.3; 3.4; 0.3**
Twist 2y −0.25, 5y 0.00, 10y +0.50, 30y +0.50: 2y; 10y; 30y	−0.50 × (−0.25%); −3.40 × 0.50%; −0.30 × 0.50%	**+0.125%; −1.700%; −0.150%**
Total change in value		**−1.725% (parallel +0.5 would cost 2.75%)**

5. Empirical duration

- Analytical duration comes from a formula or model and holds the spread constant. Empirical duration is measured from real price history: regress the bond's price changes on benchmark yield changes; the slope, with its sign flipped, is the empirical duration.
- It differs most for lower-quality credit. In a flight to quality, government yields fall but the spread on risky bonds widens at the same time, so the bond's own yield can rise and its price fall.
- Hedging with analytical duration could leave the fund under-protected.

HIGH-YIELD BOND, ANALYTICAL DURATION 5.0

Government yield −40bp: model prediction	5.0 × 0.40%	**+2.0%**
Spread +90bp: bond's own yield change	−40 + 90 bp	**+50bp**
Actual price change	−5.0 × 0.50%	**−2.5%**
Regression slope on four observations	−0.02% per bp, i.e. −2% per point	**empirical duration 2.0 (vs 5.0)**

6. Investment horizon and the duration gap

- After a rise in rates the price falls but coupons are reinvested at higher rates; after a fall the price rises but reinvestment earns less. The effects offset, and the balance depends on how long you hold.
- For a single parallel shift soon after purchase, when the horizon equals Macaulay duration the effects cancel and the realised return is close to the yield bought at (the idea behind immunisation).
- Duration gap = Macaulay duration − horizon. Zero: matched. Positive: price risk dominates, rising rates hurt. Negative: reinvestment risk dominates, falling rates hurt.

Duration gap = Macaulay duration − investment horizon

ALDER BOND, MACAULAY DURATION 8.34 YEARS (833.92 ÷ 100.00; MODIFIED 8.34 ÷ 1.02 = 8.18)

Horizon 5 years: gap	8.34 − 5.00	**+3.34, price risk, rates rise**
Horizon 10 years: gap	8.34 − 10.00	**−1.66, reinvestment risk, rates fall**
Bought at par, yields jump to 5%: return at 5 years		**3.34%**
At 7 years; 8.5 years; 10 years		**3.82%; 4.02%; 4.17%**

Key takeaways

- Convexity adds ½ × C × Δy² to the duration estimate; it is positive for option-free bonds.
- Portfolio duration is the market-value-weighted average and fits only a parallel shift.
- Bonds with options need effective duration, from a pricing model, not modified duration.
- Key rate durations show where on the curve the risk sits and sum to about the effective duration.
- Empirical duration, measured from history, can differ from analytical duration for lower-quality credit.
- Duration gap = Macaulay duration − horizon: positive means rising rates hurt, negative means falling rates hurt.

Quick check

1. Modified duration 7, convexity 70. Yields rise 1%. Estimated price change?
   Answer: −7.00% + 0.35% = −6.65%.
2. 60% in duration 4, 40% in duration 9. Portfolio duration?
   Answer: 0.6 × 4 + 0.4 × 9 = 6.0.
3. Macaulay duration 6 years, horizon 8 years. Gap and exposure?
   Answer: Gap −2 years. Falling rates hurt (reinvestment risk).