CHARTA LEARNING · LESSON NOTES

Pricing Financial Forwards and Futures

FRM Part I

1 A story: a price fixed today

- Agree a price today, trade at a fixed future date
- That agreement is a forward contract
- The seller could instead buy now, borrow, and hold
- So the forward price is a replication cost

IN THE LESSON

- Imagine a bakery that will need a large order of flour in six months.
- The baker worries the price may jump, so she signs an agreement today to buy at a fixed price on a fixed date.
- That agreement is a forward contract.
- Now ask what fixed price is fair.
- You might guess the seller sets it by predicting next season's price.
- In fact, for a financial asset, the seller can just buy the asset today, borrow money to pay for it, and hold it until delivery.
- The price of that route sets the price of the promise.
- Keep this story in mind: the forward price is a replication cost, not a prediction.

2 What is a financial asset?

- A claim on future cash flows, or a stake in an entity
- Examples: share, bond, currency deposit, stock index
- Value comes from the payments, not from physical use
- No storage cost and no convenience yield
- So arbitrage pins the forward price to spot

IN THE LESSON

- A financial asset is a claim on future cash flows, or a stake in some entity.
- Think of a share, a bond, a currency deposit, or a stock index.
- Its value comes from the payments it entitles the holder to, not from any physical use.
- This matters for forwards because financial assets can be priced by arbitrage.
- An investor can buy the asset, or sell it short, and take the opposite forward position, so the forward price is pinned to the spot price, the financing rate and any known income.
- A financial asset has no storage cost and no convenience yield.
- That is what makes the pricing exact rather than just a range.

3 Investment assets and consumption commodities

Which assets can arbitrage price exactly?

Asset type	Forward price by arbitrage
Share, bond, index	Exact price
Currency deposit	Exact price
Gold held for investment	Exact price
Oil, copper (consumption)	Only bounds

IN THE LESSON

- Not every asset can be pinned down so tightly.
- Investment assets, such as shares, bonds and gold held by many investors purely as an investment, can be priced by arbitrage, because holders will buy or sell if the forward price drifts away from the replication value.
- Consumption commodities, such as oil or copper, are different.
- Physical holders pay storage costs, and they enjoy a convenience yield from having the real thing on hand.
- Because nobody can easily sell the physical stock short and buy it back later, arbitrage gives only bounds, not a single exact price.
- This lesson focuses on the first group, the financial assets.

4 What short selling means

- Borrow shares, sell them now, buy back later and return
- Profit if the price falls; loss if it rises
- Loss is unlimited, because a price can keep rising
- A dividend must be paid over to the lender

IN THE LESSON

- Replication often needs a short sale, so let's see how one works.
- Short selling means selling an asset you do not own.
- The seller borrows the shares from a lender, sells them in the market, and later buys the same number of shares back to return to the lender.
- The short seller profits if the price falls and loses if it rises.
- Because a price can rise without limit, so can the potential loss.
- And if the stock pays a dividend while the position is open, the short seller must pay the lender an amount equal to that dividend, so the lender is left as well off as if the shares had never been lent.

5 Net profit on a short sale

Net profit = Sale − Repurchase − Dividends
Break-even price = Sale − Dividends

Net profit	Result per share
Sale price	Price received when selling
Repurchase price	Price paid to return the shares
Dividends paid	Paid to the lender, so subtracted

IN THE LESSON

- Ignoring borrowing fees and interest on margin, the result per share comes down to one line.
- Net profit equals the sale price, minus the repurchase price, minus any dividends paid to the lender.
- If you sell high and buy back low you gain, but every dividend you must pay reduces the result.
- Setting the profit to zero gives the break even point, the repurchase price at which the seller neither gains nor loses.
- The break even price equals the sale price minus the dividends paid.
- The usual slip is to ignore the dividend, or even to add it, as though the short seller received it.
- The short seller pays it.

6 Example 1: shorting a dividend payer

Illustration: sell short one share; per share figures

Sale price		**$50.00**
Dividend paid to lender		**$1.50**
Repurchase price		**$45.00**
Net profit per share	50 − 45 − 1.50	**$3.50**
Break-even price	50 − 1.50	**$48.50**

IN THE LESSON

- Here is an illustration with round numbers.
- You borrow a share and sell it at fifty dollars.
- While the position is open the company pays a dividend of one dollar fifty, which you must pass to the lender.
- Later the price has fallen to forty five dollars, and you buy the share back.
- Net profit is fifty, minus forty five, minus one fifty, which is three dollars fifty.
- The break even price is fifty minus one fifty, so forty eight dollars fifty.
- Notice that you would make money at a repurchase price of forty eight, but would lose at forty nine, even though both are below the sale price.
- The dividend is why.

7 Forwards versus futures

- Both fix a price today for a trade at a future date
- Forward: private, customized, settled once, credit risk
- Futures: exchange, standard, daily marking, margin
- Most futures are closed out before delivery

IN THE LESSON

- Now the contracts themselves.
- Forward and futures contracts both fix a price today for buying or selling an asset at a future date.
- A forward is a private agreement made over the counter.
- It is customized in amount and date, settled once at maturity, and exposed to the credit risk of the counterparty.
- A futures contract is standardized and traded on an exchange.
- A clearinghouse guarantees it, and it is marked to market every day, with margin posted, so gains and losses are paid each day rather than at the end.
- Most futures contracts are closed out before delivery, by taking the opposite position, rather than by delivering the asset.

8 Forward and futures side by side

Same economic idea, different trading arrangements

Feature	Forward | Futures
Where traded	OTC | Exchange
Terms	Custom | Standard
Settlement	At maturity | Daily
Credit risk	Counterparty | Clearinghouse
How it ends	Delivery | Closed out

IN THE LESSON

- Let's put the differences in a table.
- On where it trades, a forward is over the counter, while a futures contract trades on an exchange.
- On terms, the forward is customized and the futures contract is standardized.
- On settlement, the forward settles once, at maturity, while futures settle every day through marking to market.
- On credit risk, the forward carries the risk that the counterparty fails, while the futures clearinghouse guarantees performance and margin protects it.
- And on how they end, forwards usually run to delivery, while most futures are closed out early.
- The pricing ideas in this lesson apply to both.

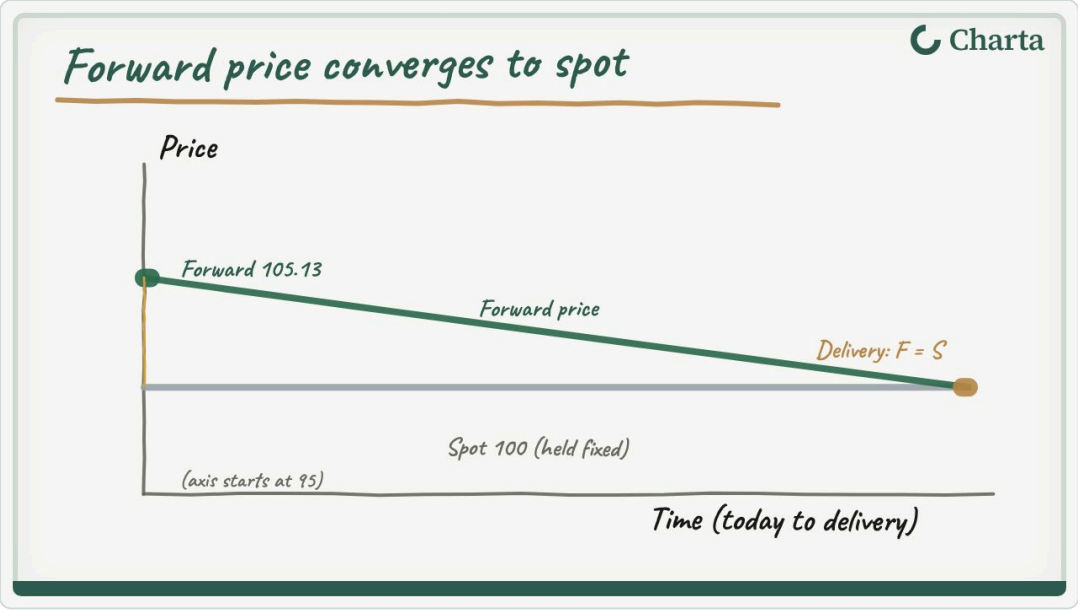
9 How the forward price relates to spot

- Forward price: makes a new contract worth zero
- Forward above spot if financing cost exceeds income
- Forward below spot if income exceeds financing cost
- It converges to spot at maturity
- It is not a forecast of the future spot price

IN THE LESSON

- The forward price is the delivery price that makes a new contract worth zero today.
- It is linked to spot through the cost of carry.
- The cost of carry is the financing cost of holding the asset, minus any income it pays.
- So the forward price exceeds spot when financing costs exceed the income, and lies below spot when the income exceeds financing costs.
- As the contract approaches maturity, the forward price converges to spot, because on the delivery date the two are the same trade.
- Finally, the forward price is not a forecast of the future spot price.
- It is set by the cost of replicating the position today.

10 Forward price converges to spot



IN THE LESSON

- Here is the idea as a picture.
- Along the bottom is time, from today on the left to delivery on the right.
- Up the side is price.
- To keep it simple, suppose the spot price stays at one hundred dollars.
- The flat line is spot.
- Today, with a five per cent rate and one year to go, the forward price sits above spot, at about one hundred and five dollars and thirteen cents, because financing costs exceed income.
- As time passes the forward price slides down, and at delivery it lands exactly on spot, at one hundred.
- This is convergence.
- The slide down is the cost of carry running out.
- A long who locked in the high price pays more than spot at delivery, but saved the financing cost of buying the asset today.

11 The forward price with no income

$F_0 = S_0 \times e^{rT}$
Annual compounding: $F_0 = S_0 \times (1 + r)^T$

Forward price, F	Forward price today
Spot price, S	Spot price today
Risk-free rate, r	Risk-free rate, continuous
Time to delivery, T	Years to delivery

IN THE LESSON

- For an asset with no income or carrying costs, the forward price comes from replication.
- With continuous compounding, the forward price equals the spot price times the exponential of the rate times the time.
- Here the spot price is today's price of the asset, the rate is the continuously compounded risk free rate, and the time is the years to delivery.
- In words, you compound the spot price forward at the risk free rate.
- Check the compounding convention in the question.
- With annual compounding, the forward price equals the spot price times one plus the rate, raised to the power of the time.
- Always express the time in years.

12 Example 2: a six month forward

Illustration: $S_0 = \$100$, $r = 5\%$, no income, $T = 0.5$ years

Exponent r × T	0.05 × 0.5	**0.025**
Growth factor	e^{0.025}	**1.0253**
Forward price	100 × 1.0253	**$102.53**
Annual compounding	100 × 1.05^{0.5}	**$102.47**

IN THE LESSON

- Illustration.
- A share that pays no dividends is priced at one hundred dollars.
- The continuously compounded risk free rate is five per cent, and delivery is in six months, so the time is one half of a year.
- The exponent is five per cent times one half, which is point zero two five.
- The exponential of that is about one point zero two five three.
- Multiply by one hundred and the forward price is one hundred and two dollars fifty three.
- If the question instead used annual compounding, we would raise one point zero five to the power one half, giving one hundred and two dollars forty seven.
- Same idea, slightly different answer, so check the convention.

13 Why arbitrage enforces the price

- Forward too high: borrow, buy asset, sell forward
- That is cash and carry
- Forward too low: short asset, lend, buy forward
- That is reverse cash and carry
- No outlay, no risk: so arbitrage forces F to fair value

IN THE LESSON

- Why must the forward price take that value?
- Because otherwise there is a free lunch.
- Suppose the market forward price is higher than the replication value.
- An arbitrageur borrows at the risk free rate, buys the asset, and sells the forward.
- At maturity the asset is delivered at the high forward price, the loan is repaid, and the difference is kept.
- This trade is called cash and carry.
- Now suppose the forward price is lower.
- The arbitrageur sells the asset short, lends the proceeds, and buys the forward.
- That is reverse cash and carry.
- Either trade needs no initial outlay and carries no risk, so competition pushes the forward price back to the fair value.

14 Example 3: cash and carry

Illustration: $S_0 = \$100$, $r = 5\%$, $T = 1$ year. Market forward is $108

Fair forward price	100 × e^{0.05}	**$105.13**
Receive at delivery	forward sold at 108	**$108.00**
Repay loan with interest	100 × e^{0.05}	**$105.13**
Riskless profit	108 − 105.13	**$2.87**
Initial outlay	borrowed all of it	**$0**

IN THE LESSON

- Illustration.
- The spot price is one hundred dollars, the rate is five per cent, and the time is one year.
- The fair forward price is one hundred times the exponential of point zero five, which is one hundred and five dollars thirteen.
- But suppose the market forward price is one hundred and eight.
- You borrow one hundred dollars, buy the asset, and sell the forward at one hundred and eight.
- At maturity you deliver the asset and receive one hundred and eight.
- You repay the loan with interest, which is one hundred and five thirteen.
- What remains is a riskless profit of two dollars eighty seven, with nothing invested at the start.

15 Both arbitrage trades at a glance

Illustration: fair forward price is $105.13

Market forward price	Action and result
$108 (too high)	Cash and carry
Profit at delivery	$108 − $105.13 = $2.87
$103 (too low)	Reverse cash and carry
Profit at delivery	$105.13 − $103 = $2.13

IN THE LESSON

- Let's compare the two trades using the same numbers.
- If the market forward price is one hundred and eight, above the fair one hundred and five thirteen, we run cash and carry.
- We borrow, buy the asset and sell the forward, and lock in two dollars eighty seven.
- If instead the market forward is one hundred and three, below fair value, we run reverse cash and carry.
- We short the asset, lend the one hundred dollars, which grows to one hundred and five thirteen, and buy the forward for one hundred and three.
- At delivery we take the asset to close the short, and keep two dollars thirteen.
- Either way the gap is captured.

16 The value of a forward contract

f = (F − K) × e^{−rT}
No income: f = S − K × e^{−rT}

Contract value, f	Value of the long forward today
Forward price, F	Current forward price
Delivery price, K	Delivery price in the contract
Risk-free rate, r	Risk-free rate
Time to delivery, T	Time left to delivery

IN THE LESSON

- Now a key distinction.
- The forward price is not the same thing as the value of a forward contract.
- The delivery price, called K, is the fixed price written into an existing contract.
- At initiation K equals the forward price, and the value is zero.
- Later, with time T left to delivery, the market forward price F for the remaining life may have moved.
- The long forward is then worth F minus K, discounted back at the risk free rate.
- For an asset with no income, this equals the spot price minus the present value of K.
- The short position is worth the negative of this.

17 Example 4: valuing a contract later

Illustration: K = $105.13. Six months later S = $108, r = 5%, T left = 0.5

Delivery price K	set at initiation	**$105.13**
New forward price F	108 × e^{0.025}	**$110.73**
Discounted difference	(F − K) × e^{−0.025}	**$5.47**
Shortcut check	108 − 105.13 × e^{−0.025}	**$5.47**

IN THE LESSON

- Illustration, continuing example three.
- We entered a one year forward on a share worth one hundred dollars, so the delivery price is one hundred and five dollars thirteen.
- Six months later the share is worth one hundred and eight dollars.
- With half a year left, the new forward price is one hundred and eight times the exponential of point zero two five, which is one hundred and ten dollars seventy three.
- The long is worth the difference from the delivery price, about five dollars sixty, discounted, which gives five dollars forty seven.
- Check the shortcut: spot minus the present value of the delivery price is one hundred and eight, minus one hundred and two fifty three, also five forty seven.

18 Price versus value

- Forward price: makes a NEW contract worth zero today
- Value: what YOUR existing contract is worth now
- At initiation K = F, so the value is zero
- Value can be negative; a share price cannot
- Do not forget to discount F − K

IN THE LESSON

- Let's make the contrast sharp.
- The forward price is the delivery price that would make a new contract worth zero today, and it changes as spot prices and rates change.
- The value of a contract is what your existing position is worth, and it can be positive or negative.
- At initiation the value is zero, even though the forward price is one hundred and five.
- Typical mistakes are to report F minus K without discounting, or to treat the forward price as the value of the contract.
- Notice too that a contract's value can be negative, whereas the forward price of a share, like any share price, is positive.

19 Forward price when the asset pays income

Known income: $F_0 = (S_0 - I) \times e^{rT}$
Yield q: $F_0 = S_0 e^{(r-q)T}$, $f = S_0 e^{-qT} - K \cdot e^{-rT}$

Forward price, F	Forward price today
Income, I	Present value of income
Yield, q	Continuous yield on the asset
Contract value, f	Long value: $S_0 - I - K \cdot e^{-rT}$

IN THE LESSON

- Now the case where the asset pays something while you wait.
- Income is a benefit the forward holder misses, because the holder does not own the asset until delivery.
- So income lowers the forward price.
- If the income over the life of the contract has a present value I, the forward price equals spot minus I, all compounded at the risk free rate.
- A long forward with delivery price K is worth spot, minus I, minus the present value of K.
- For an asset paying a continuous yield q, such as an index's dividend yield or a currency's foreign rate, the forward price equals spot times the exponential of r minus q, times T.
- With a yield, the long forward is worth spot times e to the minus q T, minus K times e to the minus r T.
- Setting income to zero gives back the no income formula.

20 Example 5: a dividend before delivery

Illustration: $S_0 = \$100$, $r = 4\%$, $T = 1$. Dividend \$3 paid at 6 months

PV of the dividend I	$3 \times e^{-0.02}$	**\$2.94**
Spot less income	$100 - 2.94$	**\$97.06**
Forward price	$97.06 \times e^{0.04}$	**\$101.02**
Without the dividend	$100 \times e^{0.04}$	**\$104.08**

IN THE LESSON

- Illustration.
- A share is priced at one hundred dollars, the rate is four per cent, and the forward is for one year.
- The share will pay a three dollar dividend in six months.
- First find the present value of that dividend.
- Discount it for half a year at four per cent: three times the exponential of minus point zero two, which is two dollars ninety four.
- Subtract it from spot, leaving ninety seven dollars six.
- Now compound that for the full year.
- The forward price is about one hundred and one dollars two cents.
- Without the dividend it would have been one hundred and four dollars eight cents.
- Discount each dividend over its own time to payment, not the full year.

21 Example 6: a currency forward

Illustration: S = 1.2000, r = 5%, q = 2%, T = 1 year

Net carry r − q	5% − 2%	**3%**
Forward price	1.20 × e^{0.03}	**1.2365**
Spot discounted at q	1.20 × e^{−0.02}	**1.1762**
Long value, K = 1.20	1.1762 − 1.20 × e^{−0.05}	**0.0348**

IN THE LESSON

- Illustration with a currency.
- Spot is one dollar twenty per unit of foreign currency.
- The domestic rate is five per cent, and the foreign rate, which acts as the yield, is two per cent.
- The time is one year.
- The forward price is spot times the exponential of five minus two per cent, so of point zero three.
- That gives about one point two three six five dollars.
- The forward is higher than spot because the domestic rate is higher than the foreign yield.
- Now value an old contract with a delivery price of one dollar twenty.
- It is worth spot times e to the minus foreign rate, minus the present value of the delivery price, about three point four eight cents.

22 Forward prices versus futures prices

- Equal if rates are constant and the same for all terms
- Difference comes from daily settlement of futures
- Positive correlation with rates: futures above forward
- Negative correlation with rates: futures below forward

IN THE LESSON

- Are forward and futures prices the same?
- For the same asset and maturity they are equal when interest rates are constant and the same for all maturities.
- The difference in practice arises because futures are settled daily, so gains and losses are reinvested or financed along the way.
- If the asset price is positively correlated with interest rates, a long futures holder tends to gain when rates are high, and so reinvests at high rates, and tends to lose when rates are low, and so funds at low rates.
- Futures are then more attractive and the futures price is above the forward price.
- With negative correlation the futures price is below the forward price.

23 When the gap matters

Futures price compared with forward price

Situation	Futures vs forward
Constant rates or zero correlation	Equal
Positive correlation with rates	Futures higher
Negative correlation with rates	Futures lower
Short maturity, in practice	Gap tiny: treat as equal

IN THE LESSON

- How big is the gap in practice?
- For contracts of a few months it is usually too small to matter next to transaction costs, margin requirements and taxes.
- But the gap grows with maturity, so for long dated contracts it can become noticeable.
- In calculations, unless the question says otherwise, treat the two prices as equal.
- The table sums up the cases.
- With constant rates, or no link between the asset and rates, the two are equal.
- With positive correlation, the futures price is higher.
- With negative correlation, the futures price is lower.
- The effect comes from reinvesting daily gains, not from any change in the asset's value.

24 Stock index futures

F = S × e^{(r − q)T}
Contract value = Multiplier × F

Futures price, F	Futures price
Spot price, S	Index level
Yield, q	Dividend yield of the index
Multiplier	Dollars per index point

IN THE LESSON

- A stock index is treated as an asset paying a continuous dividend yield, q.
- So the futures price equals the index level times the exponential of r minus q, times T, where r is the risk free rate and T is the years to expiry.
- The notional value of one contract is the multiplier times the futures price.
- The usual mistake is to ignore q.
- When the dividend yield is above the risk free rate, the futures price is actually below the index level.
- That is normal, and should not be read as an arbitrage.
- It simply says the income from the stocks outweighs the cost of financing them.

25 Example 7: pricing an index future

Illustration: S = 4000, r = 5%, q = 2%, T = 0.5, multiplier $50

Exponent (r − q) × T	0.03 × 0.5	**0.015**
Fair futures price	4000 × e^{0.015}	**4060.45**
Multiplier	dollars per point	**$50**
Notional of one contract	50 × 4060.45	**$203,023**

IN THE LESSON

- Illustration.
- An index stands at four thousand.
- The risk free rate is five per cent, the index dividend yield is two per cent, and the contract expires in six months.
- The net carry is five minus two, which is three per cent a year.
- Over half a year the exponent is point zero one five.
- The futures price is four thousand times the exponential of that, about four thousand sixty point four five.
- Suppose the contract multiplier is fifty dollars per index point.
- The notional value of one contract is fifty times four thousand sixty point four five, which is about two hundred three thousand dollars.
- The multiplier is a made up number for this illustration.

26 How index arbitrage works

- Futures too high: sell futures, buy the stocks, borrow
- Futures too low: buy futures, short the stocks, invest
- Program trading moves the whole basket at once
- Costs and tracking error create a no-arbitrage band

IN THE LESSON

- If the futures price is above its theoretical value, index arbitrage sells futures and buys the stocks in the index, financed by borrowing.
- If it is below, the arbitrageur buys futures and sells the stocks short, investing the proceeds.
- In practice this is done by program trading, which buys or sells the whole basket of stocks at once, rather than one name at a time.
- Transaction costs and imperfect tracking create a band around the theoretical value, within which arbitrage does not pay.
- Only when the futures price moves outside that band does the trade become attractive.
- This is the same cash and carry logic we met earlier, applied to a basket.

27 Example 8: an index arbitrage

Illustration: fair 4060.45, market futures 4100, multiplier $50

Fair futures price	4000 × e^{0.015}	**4060.45**
Market futures price	above fair value	**4100.00**
Gap in index points	4100 − 4060.45	**39.55**
Trade, financed by borrowing	sell futures, buy stocks	—
Gain per contract	39.548 × 50, before costs	≈ **$1,977**

IN THE LESSON

- Illustration, using the same index.
- The fair futures price is four thousand sixty point four five.
- Suppose the market futures price is four thousand one hundred, above fair value.
- The gap is thirty nine point five five index points.
- The arbitrageur sells futures and buys the basket of stocks in the index, financed by borrowing.
- Holding the stocks earns their dividends, and borrowing costs interest, and at expiry the futures position settles against the index.
- The result is a locked in gain of the gap, thirty nine point five five points.
- With a multiplier of fifty, that is about nineteen hundred seventy seven dollars per contract, before costs.

28 Four mistakes to avoid

- Forward is a replication cost, not a spot forecast
- Value ≠ price: discount F − K, do not report F alone
- Short sale: subtract dividends paid, never add them
- Income: subtract its PV (or use q); check compounding

IN THE LESSON

- Mistake one: reading the forward price as a forecast of the future spot price.
- It is a replication cost, set by spot, the financing rate and income.
- Mistake two: confusing price with value, or forgetting to discount.
- An existing contract is worth the difference between F and K, discounted.
- Mistake three: getting short sales wrong.
- The short seller pays the dividend, so it is subtracted from profit, never added.
- Mistake four: mishandling income and conventions.
- Subtract the present value of income from spot, or use the yield q, and check whether compounding is continuous or annual, with time in years.

29 Recap: the main ideas

- Asset = claim on cash flows; short pays dividends
- $F_0 = S_0e^{rT}$; income lowers it; arbitrage enforces it
- Value f = (F − K)e^{−rT}; price is not value
- Futures ≈ forward unless told otherwise
- Index: F = S·e^{(r−q)T}; program trading arbitrage

IN THE LESSON

- Let's pull it together.
- A financial asset is a claim on cash flows, so arbitrage prices its forwards exactly.
- A short seller borrows, sells, and buys back, and pays any dividends.
- Forwards are private and settled once, futures are exchange traded and settled daily.
- The forward price is spot compounded at the risk free rate, less the present value of income, and it converges to spot.
- Cash and carry and reverse cash and carry enforce it.
- A contract's value is the discounted difference between the forward price and the delivery price.
- Futures and forwards are treated as equal.
- An index is an asset with a yield.

30 Try three quick questions

1. Short at $60, dividend $2, buy back at $55. Net profit?
   Answer: 60 − 55 − 2 = $3 per share.
2. S = $80, no income, r = 6%, T = 2. Forward price?
   Answer: 80 × e^{0.12} = $90.20 (not discounted).
3. Index 2000, r = 4%, q = 3%, T = 1. Futures price?
   Answer: 2000 × e^{0.01} = 2020.10.

IN THE LESSON

- Now try three quick questions yourself.
- Pause after each one.
- Question one.
- A share is sold short at sixty dollars, pays a two dollar dividend, and is bought back at fifty five.
- What is the net profit per share?
- Sixty minus fifty five minus two is three dollars.
- Question two.
- A share with no income costs eighty dollars, the continuous rate is six per cent, and delivery is in two years.
- What is the forward price?
- Eighty times the exponential of point one two, about ninety dollars twenty.
- Question three.
- An index is at two thousand, the rate is four per cent, the yield is three per cent, and expiry is in one year.
- What is the futures price?
- About two thousand twenty.
- The full answers are on the board.