CHARTA LEARNING · LESSON NOTES

Properties of Interest Rates

FRM Part I

1 A rate is never just a number

- '8%' means little until you know the convention
- Which borrower? Government, bank, or a secured loan
- How often is interest added: yearly, monthly, always?
- Over what horizon: one night, one year, ten years?

IN THE LESSON

- Imagine three friends who each say they earn eight per cent.
- One lent to a government, one lent to a bank, and one lent for a single night against collateral.
- Their eights are not the same thing.
- Even with the same borrower, one friend may be paid once a year and another every month, and the monthly friend ends up with more money.
- And a rate for one night says little about a rate for ten years.
- So before using any rate we ask four questions.
- Who is borrowing?
- How often is interest added?
- Over what horizon?
- And how do rates for different horizons relate to one another?
- This lesson answers them in that order.

2 Treasury rates

- Yield on US government debt: bills, notes, bonds
- Bills run to one year; notes and bonds go longer
- Nearly default-free, but not a clean risk-free proxy
- Regulation, liquidity needs, state tax exemption
- These push Treasury yields below other low-risk rates

IN THE LESSON

- Start with the rate most people think of as the safe one.
- A Treasury rate is the yield on debt issued by the United States government.
- Bills cover maturities up to one year, and notes and bonds cover the longer ones.
- Because the government is so unlikely to default, Treasuries are nearly default free.
- But that does not make the Treasury yield a clean risk free proxy.
- Regulation, the liquidity needs of institutions, and the state tax exemption on Treasury interest all create extra demand for Treasuries.
- That demand pushes their yields below other low risk rates.
- So the Treasury rate is the safest rate, but it is also a little too low to stand for pure compensation for time alone.

3 Repo: a loan in disguise

- Sell securities now, agree to buy them back higher
- Economically a collateralized loan
- Securities = collateral; price difference = interest
- Repo rate = price difference ÷ cash lent, annualized
- Usually on an actual/360 day count

IN THE LESSON

- Next, the repo.
- A repo is a sale of securities together with an agreement to buy them back later at a higher price.
- Look at it from the cash side and it is simply a loan.
- The securities act as collateral, and the extra amount paid at repurchase is the interest.
- The repo rate is that price difference, divided by the cash lent, and then annualized.
- Day counts matter here: repo is usually quoted on actual over three hundred and sixty, meaning the real number of days held, against a year of three hundred and sixty days.
- Because a lender holds good collateral, repo rates tend to be low.
- Let's put numbers on one overnight trade.

4 Example 1: a repo rate

Illustration: borrow cash overnight against Treasuries

Cash lent today		**10,000,000**
Repurchase price tomorrow		**10,001,250**
Interest for the night	10,001,250 − 10,000,000	**1,250**
Repo rate	1,250 ÷ 10M × 360 ÷ 1	**4.50%**

IN THE LESSON

- Here is an overnight repo with made up numbers.
- A dealer sells Treasuries for ten million dollars in cash, and agrees to buy them back tomorrow.
- The repurchase price is ten million, one thousand, two hundred and fifty dollars.
- So the interest for the night is the difference: one thousand two hundred and fifty dollars.
- Now annualize.
- Divide the interest by the cash lent, which gives a very small fraction.
- Multiply by three hundred and sixty, because of the day count, and divide by the one day the money was out.
- The repo rate is four point five per cent.
- Notice that the dealer is the borrower and the securities are only collateral.
- That safety is why this rate is low.

5 SOFR, and what risk-free means

- SOFR: overnight Treasury-backed repo rate, NY Fed
- Secured: almost no bank credit premium
- It replaced unsecured LIBOR as the benchmark
- Mistake: calling SOFR an unsecured interbank rate
- Risk-free is a proxy, chosen by purpose

IN THE LESSON

- Now S O F R, the Secured Overnight Financing Rate.
- It is the overnight rate on repo trades that are collateralized by Treasuries, and it is published by the New York Fed.
- Because every trade is secured, it carries almost no bank credit premium.
- That is exactly why it replaced Libor, which was an unsecured rate, as the benchmark.
- The usual mistake is to treat S O F R as an unsecured interbank rate.
- It is not.
- So what does risk free mean?
- No rate is perfectly free of risk.
- The risk free rate is a proxy that we choose for the job in hand, such as a Treasury yield or S O F R, knowing each has its own small quirks.

6 Compounding frequency

Terminal value = A × (1 + R/m)^(mT)
Continuous: A × e^(RT)

Amount invested, A	Amount invested today
Quoted rate, R	Quoted annual rate
Periods per year, m	Compounding periods per year
Years, T	Years invested

IN THE LESSON

- Now to the arithmetic of growth.
- Suppose you invest an amount for some years at a quoted annual rate, with interest added a number of times each year.
- The terminal value is the amount times one plus the rate divided by the number of periods, all raised to the power of the number of periods per year times the years.
- The more often interest is added, the more interest earns interest.
- Push the frequency to the limit and interest is added at every instant.
- That is continuous compounding, and the terminal value becomes the amount times e raised to the power of the rate times the years.
- A common error is counting periods wrongly, using years where months are needed.

7 Example 2: 100 at 8% for one year

Illustration: invest 100 at a quoted 8% for 1 year

Annual (m = 1)	100 × 1.08	**108.00**
Semiannual (m = 2)	100 × 1.04^2	**108.16**
Quarterly (m = 4)	100 × 1.02^4	**108.24**
Monthly (m = 12)	100 × (1 + 0.08/12)^12	**108.30**
Continuous	100 × e^{0.08}	**108.33**

IN THE LESSON

- Let's see the effect with a made up investment of one hundred at a quoted eight per cent for one year.
- Compounded annually, we get one hundred and eight.
- Compounded twice a year, each half year earns four per cent, so we multiply by one point zero four, twice, and reach one hundred and eight sixteen.
- Quarterly, each quarter earns two per cent, four times, giving one hundred and eight twenty four.
- Monthly gives one hundred and eight thirty.
- Continuous compounding gives one hundred times e to the point zero eight, which is one hundred and eight thirty three.
- Notice the pattern.
- More frequent compounding always helps, but each step adds less than the one before.

8 Converting between quotes

Rc = m × ln(1 + Rm/m)
Rm = m × (e^{Rc/m} − 1)
EAR = (1 + R/m)^{m} − 1

Continuous rate, Rc	Continuous equivalent
Rate compounded m times, Rm	Rate compounded m times a year
Effective annual rate	One year's growth, annualized

IN THE LESSON

- Two quotes can be compared only at the same frequency, so we need conversions.
- To turn a rate compounded m times a year into a continuous rate, take a natural log: the continuous rate equals m times the log of one plus the rate over m.
- To go the other way, use the exponential: the rate for m periods equals m times e to the continuous rate over m, minus one.
- And the effective annual rate, E A R, is one year's growth expressed as an annual rate: one plus the rate over m, to the power m, minus one.
- If someone quotes you two rates at different frequencies, convert both to one frequency before you decide which is better.

9 Example 3: converting a quote

Illustration: an 8% rate compounded semiannually

Continuous equivalent	2 × ln(1.04)	**7.84%**
Effective annual rate	$1.04^2 - 1$	**8.16%**
Continuous 8% as annual	e^{0.08} − 1	**8.33%**
Continuous 8% as semiannual	2 × (e^{0.04} − 1)	**8.16%**

IN THE LESSON

- Take a rate of eight per cent compounded semiannually.
- First the continuous equivalent.
- Half of eight is four, so we take two times the natural log of one point zero four, which is seven point eight four per cent.
- Next the effective annual rate: one point zero four squared, minus one, is eight point one six per cent.
- That is the real annual growth.
- Now go the other direction with a continuous rate of eight per cent.
- Its annual equivalent is e to the point zero eight minus one, which is eight point three three per cent.
- And as a semiannual rate, it is two times e to the point zero four minus one, which is eight point one six per cent.

10 Bond price from its yield

Price = Σ C ÷ (1 + y)^t + F ÷ (1 + y)^n
Continuous: Σ C × e^(−yt) + F × e^(−yn)

Coupon per period	Coupon per period
Face value	Face value (principal)
Yield per period	Yield per period
Number of periods	Number of periods

IN THE LESSON

- Now bonds.
- A bond promises coupons on set dates and its face value at the end.
- Its value is the present value of all those payments, discounted at its yield.
- For each period, divide the coupon by one plus the yield per period, raised to the number of periods elapsed.
- Then add the face value divided by one plus the yield, raised to the total number of periods.
- For semiannual coupons, use half the annual coupon, half the annual yield and twice the number of years.
- The yield and the discount factors must share the same compounding convention.
- With a continuously compounded yield, each discount factor is e raised to minus the yield times the time.

11 Example 4: pricing a semiannual bond

Illustration: 2-year, 5% coupon, face 100, yield 6% semiannual

Coupon per period	5% × 100 ÷ 2	**2.50**
Yield and periods	6% ÷ 2; 2 × 2	**3%, 4**
PV of the four coupons	Σ 2.50 ÷ 1.03^t	**9.29**
PV of the face value	$100 \div 1.03^4$	**88.85**
Price	9.29 + 88.85	**98.14**

IN THE LESSON

- Let's price a two year bond with a made up five per cent coupon, a face value of one hundred, and a yield of six per cent compounded semiannually.
- The annual coupon is five, so each half year pays two point five zero.
- Next, halve the yield to three per cent and double the years, which gives four periods.
- Discounting the four coupons at three per cent and adding them up gives nine point two nine.
- The face value comes back at the end, and one hundred divided by one point zero three to the fourth is eighty eight eighty five.
- Add the two pieces together and the price is ninety eight fourteen.
- It is below one hundred because the yield beats the coupon.

12 Par, discount and premium

Same bond (5% coupon, 2 years), different yields

Yield (semiannual)	Price
5%, equal to the coupon	100.00 (par)
6%, above the coupon	98.14 (discount)
4%, below the coupon	101.90 (premium)
Rule	Yield up, price down

IN THE LESSON

- Run the same bond at other yields and a clear rule appears.
- When the yield equals the coupon rate, five per cent, the price is exactly one hundred: the bond trades at par.
- When the yield is higher than the coupon, six per cent, the price falls to ninety eight fourteen: a discount, because buyers need a better return than the coupon offers.
- When the yield is lower, four per cent, the price rises to one hundred and one ninety: a premium.
- So yield and price move in opposite directions.
- The most common errors here are using the annual yield for semiannual periods, forgetting to discount the coupons, and mixing compounding conventions.

13 A bond is a bundle of zero-coupon bonds

- Each coupon and the principal is a zero-coupon bond
- Discount each cash flow at the spot rate for its date
- Price = Σ CF(t) ÷ (1 + z(t))^t, or Σ CF(t) × e^(−z(t)t)
- YTM is a single average rate, backed out of the price
- Mistake: one spot rate for all flows; mixed compounding

IN THE LESSON

- Here is a deeper idea.
- A coupon bond is really a bundle of zero coupon bonds, one for each coupon and one for the principal.
- Each piece pays a single amount on a single date.
- So the theoretical price discounts each cash flow at the spot rate for its own maturity, also called the zero rate.
- With annual compounding, divide each cash flow by one plus its own spot rate, raised to its own time.
- With continuous rates, multiply by e to the minus the spot rate times the time.
- This differs from discounting everything at the yield to maturity, which is one single average rate backed out of the price.
- The usual errors are using one spot rate for every cash flow and mismatching the compounding.

14 Example 5: pricing with spot rates

Illustration: 3-year, 6% annual coupon. Spots 3%, 4%, 5%

Year 1: 6 at 3%	6 ÷ 1.03	**5.83**
Year 2: 6 at 4%	6 ÷ 1.04^2	**5.55**
Year 3: 106 at 5%	106 ÷ 1.05^3	**91.57**
Spot-rate price	sum of the three	**102.94**
Single yield, same price	solve for the yield	**4.92%**

IN THE LESSON

- Now an example with made up spot rates: three per cent for one year, four per cent for two years, and five per cent for three years.
- The bond pays a six per cent annual coupon for three years, with a face value of one hundred.
- The year one payment of six is discounted at three per cent, giving five eighty three.
- The year two payment of six is discounted at four per cent, over two years, giving five fifty five.
- The final payment is one hundred and six, discounted at five per cent over three years, giving ninety one fifty seven.
- The three pieces add up to a price of one hundred and two ninety four.
- A single yield that gives the same price is four point nine two per cent.

15 When price and spot value disagree

- Illustration: bond quoted at 103.50, spot value 102.94
- Rich by 0.56: sell the bond, buy its strips
- If cheaper than spot value: buy the bond, sell strips
- The gap is locked in with no risk: an arbitrage

IN THE LESSON

- What if the market price differs from the spot rate value?
- Then there is an arbitrage between the bond and its stripped cash flows.
- Suppose our bond is quoted at one hundred and three fifty, while the spot rate value is one hundred and two ninety four.
- The bond is rich by fifty six cents per hundred.
- A trader sells the bond and buys the cash flows separately as zero coupon pieces, which costs less, and pockets the gap with no risk.
- If instead the bond were cheap, the trader would buy the bond and sell the pieces.
- Trading like this pushes the two prices together, which is why the spot rate value is called the theoretical price.

16 Three duration measures

Macaulay = Σ t × PV(CF) ÷ Price
D = Macaulay ÷ (1 + y/m)
Dollar dur. = D × Price; DV01 = that × 0.0001

Macaulay duration	PV-weighted average time, years
Modified duration, D	Percent change per unit of yield
Dollar duration	Sensitivity in money terms
DV01	Loss for a one-basis-point rise

IN THE LESSON

- Duration measures how sensitive a bond's price is to its yield.
- Macaulay duration is the present value weighted average time to the bond's cash flows, in years.
- Each time is weighted by the present value of its cash flow, and the total is divided by the price.
- A zero coupon bond's equals its maturity; a coupon bond's is shorter.
- Modified duration divides Macaulay duration by one plus the yield over the compounding frequency, and gives the percentage price change per unit of yield.
- Dollar duration multiplies modified duration by the price, to express sensitivity in money.
- And the dollar value of one basis point, called D V oh one, is dollar duration times zero point zero zero zero one.

17 Example 6: durations of a 3-year bond

Illustration: 6% annual coupon, face 100, yield 5%

PV of the cash flows	5.71, 5.44, 91.57	**102.72**
Sum of t × PV	1×5.71 + 2×5.44 + 3×91.57	**291.30**
Macaulay duration	291.30 ÷ 102.72	**2.84 yrs**
Modified duration	2.84 ÷ 1.05	**2.70**

IN THE LESSON

- Take a made up three year bond with a six per cent annual coupon and a flat yield of five per cent.
- The cash flows are six, six and one hundred and six.
- Their present values are five seventy one, five forty four, and ninety one fifty seven, which add to a price of one hundred and two seventy two.
- Now weight each present value by its time: one times five seventy one, plus two times five forty four, plus three times ninety one fifty seven.
- The sum is two hundred and ninety one point three zero.
- Divide by the price and Macaulay duration is two point eight four years.
- Divide by one point zero five and modified duration is two point seven zero.

18 Example 7: dollar duration and DV01

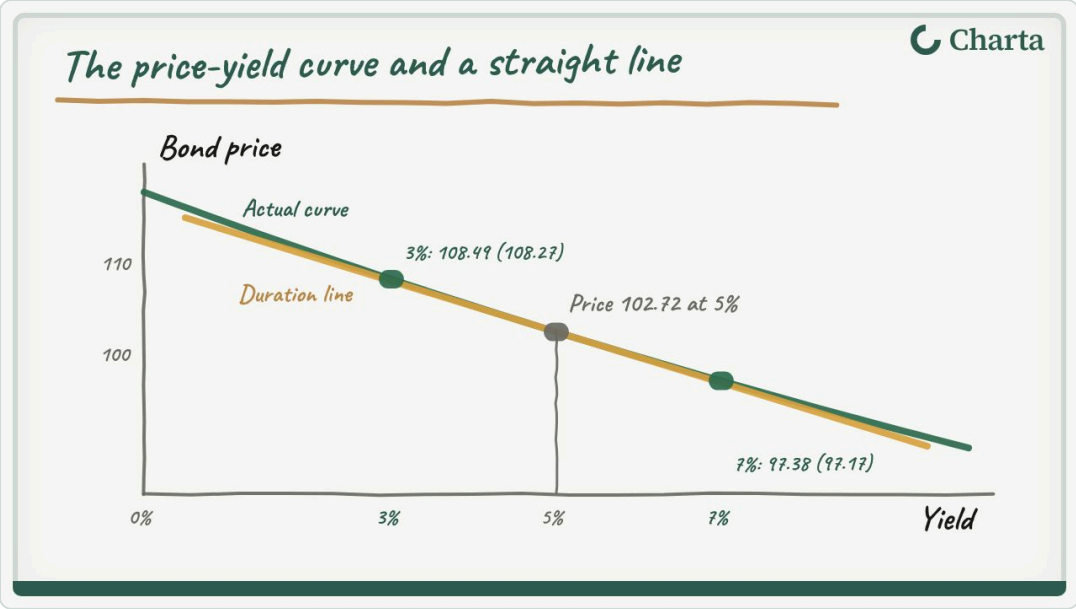
Same bond: modified duration 2.7007, price 102.72

Dollar duration	2.7007 × 102.72	**277.43**
DV01 per 1 bp rise	277.43 × 0.0001	**0.0277**
Percent change, +25 bp	−2.7007 × 0.0025	**−0.68%**
In money on 102.72	277.43 × −0.0025	**−0.69**

IN THE LESSON

- Continue with the same bond.
- Its modified duration is two point seven zero zero seven and its price is one hundred and two seventy two.
- Dollar duration is the product: two hundred and seventy seven point four three.
- Multiply by zero point zero zero zero one and a one basis point rise in yield costs about zero point zero two seven seven.
- Now suppose the yield rises by twenty five basis points.
- In decimals, that is zero point zero zero two five, never twenty five.
- The percentage change is minus two point seven zero zero seven times zero point zero zero two five, or minus zero point six eight per cent.
- On a price of one hundred and two seventy two, that is a loss of about sixty nine cents.

19 The price-yield curve and a straight line



IN THE LESSON

- Here is the picture behind duration.
- Along the bottom is the yield, and up the side is the price of our three year bond.
- The curve slopes downward: higher yield, lower price.
- And it bends, flatter on the right than on the left.
- At a yield of five per cent the price is one hundred and two seventy two.
- The straight line touching the curve there has the slope that duration describes.
- Near that point, line and curve almost agree.
- But move to seven per cent and the line gives ninety seven seventeen, while the true price is ninety seven thirty eight.
- Move to three per cent and the line gives one hundred and eight twenty seven, while the true price is one hundred and eight forty nine.
- The curve always sits above the line.

20 Where duration falls short

- Duration is the slope: a straight-line approximation
- Fails for large moves and for twists of the curve
- Convexity = curvature; zero: $T(T+1) \div (1+y)^2$
- Positive: actual price beats the estimate, up or down
- Callables and mortgage securities: negative convexity

IN THE LESSON

- Duration is the slope of the price yield curve, so it is a straight line approximation.
- That works only for small, parallel moves in yield.
- For large moves the real curve bends away from the line, and duration alone misses it.
- Duration also ignores changes in the shape of the curve, so two portfolios with equal duration can behave differently when the curve twists.
- Convexity measures the curvature: it is the second derivative of price with respect to yield, divided by price.
- For a plain bond it is positive, so the true price sits above the duration estimate whether yields rise or fall.
- For a zero coupon bond with annual compounding, convexity is the maturity times the maturity plus one, over one plus the yield squared.
- Callable bonds and mortgage securities can have negative convexity.

21 Duration plus convexity

$$\Delta P \div P \approx -D \times \Delta y + \tfrac{1}{2} \times C \times (\Delta y)^2$$

Duration term	Sign opposes the yield move
Convexity term	Positive both ways (plain bond)
Yield change	Yield change as a decimal
Modified duration	Modified, not Macaulay

IN THE LESSON

- Adding the curvature gives a second order estimate.
- The percentage price change equals minus modified duration times the yield change, plus one half times convexity times the yield change squared.
- The duration term is negative when yields rise and positive when they fall.
- The convexity term is the same sign either way, positive for a plain bond, because the yield change is squared.
- Three rules keep you safe.
- Yield changes go in as decimals, so twenty five basis points is zero point zero zero two five.
- The one half must not be forgotten, and the convexity term is added, never subtracted.
- And if you are given Macaulay duration, first divide by one plus the yield over the frequency to get modified duration.
- Multiply by the price for a money change.

22 Example 8: a 100 basis-point move

Same bond: D = 2.7007, convexity = 10.09, price 102.72

Duration term	−2.7007 × 0.01	**−2.70%**
Convexity term	½ × 10.09 × 0.01²	**+0.05%**
Estimate, yield +1%	−2.70% + 0.05%	**−2.65%**
Exact price at 6%	100.00 ÷ 102.72 − 1	**−2.65%**
Estimate, yield −1%	+2.70% + 0.05%	**+2.75%**

IN THE LESSON

- Apply this to our bond, whose modified duration is two point seven zero zero seven and convexity is ten point zero nine.
- Say the yield rises by one per cent, or zero point zero one.
- The duration term is minus two point seven zero per cent.
- The convexity term is one half times ten point zero nine times zero point zero one squared, a gain of zero point zero five per cent.
- Together the estimate is minus two point six five per cent.
- The exact price at six per cent is exactly one hundred, a change of minus two point six five per cent, so the two agree.
- Duration alone would have said minus two point seven zero.
- If yields fall one per cent instead, the estimate is plus two point seven five.

23 Forward rates from spot rates

F = [(1+R2)^T2 ÷ (1+R1)^T1]^(1÷(T2−T1)) − 1
Continuous: F = (R2×T2 − R1×T1) ÷ (T2 − T1)

Spot rate to T1	Spot rate to the earlier date
Spot rate to T2	Spot rate to the later date
Forward rate	Rate implied for T1 to T2

IN THE LESSON

- Now forward rates.
- A forward rate is the rate for a future period that today's spot curve implies, because of no arbitrage.
- Investing for T two years at the longer spot rate must give the same result as investing for T one years at the shorter spot rate, then rolling over at the forward rate.
- With annual compounding, divide the two growth factors, raise the result to one over the length of the forward period, and subtract one.
- With continuous compounding it is much simpler: the rate times time terms just subtract, and we divide by the length of the period.
- Three common errors: averaging spot rates, taking differences of rates, and forgetting to annualize over the length of the forward period.

24 Example 9: forwards from a spot curve

Illustration: annual spot rates 3% (1y), 4% (2y), 5% (3y)

Forward, year 1 to 2	1.04² ÷ 1.03 − 1	**5.01%**
Forward, year 2 to 3	1.05³ ÷ 1.04² − 1	**7.03%**
Check: 3-year growth	1.03 × 1.0501 × 1.0703	**1.1576**
Spot growth to year 3	1.05³	**1.1576**
Continuous 3%, 4%: forward	(4%×2 − 3%×1) ÷ 1	**5.00%**

IN THE LESSON

- Use made up annual spot rates: three per cent for one year, four per cent for two years, five per cent for three years.
- The forward rate from year one to year two is one point zero four squared, divided by one point zero three, minus one: five point zero one per cent.
- From year two to year three it is one point zero five cubed over one point zero four squared, minus one: seven point zero three per cent.
- Check by going the other way.
- Three year growth is one point zero three times one point zero five zero one times one point zero seven zero three, which is one point one five seven six, the same as one point zero five cubed.
- Finally, with continuous rates of three and four, the forward is four times two, minus three times one, which is five per cent.

25 Forward rate agreements

Net payment at T2 = L × (RK − RF) × (T2 − T1)
Value today = Payment × DF(T2)

Notional, L	Notional principal
Fixed rate, RK	Agreed fixed rate
Forward rate, RF	Forward rate for the period
Discount factor	Discount factor to T2
Settlement	Often settled at T1, discounted

IN THE LESSON

- A forward rate agreement, or F R A, fixes an interest rate for a future period from T one to T two, on a notional principal.
- One side receives interest at the fixed rate and pays the floating rate, or the reverse.
- If the floating rate turns out equal to the forward rate, the receiver of fixed gets a net payment at T two: the notional times the fixed rate minus the forward rate, times the length of the period.
- The value today is that payment discounted at the T two zero rate.
- At inception the fixed rate equals the forward rate, so the value is zero.
- In practice, F R As are often settled at T one, with the difference discounted back.
- Typical mistakes are discounting to the wrong date and flipping the sign.

26 Example 10: valuing a FRA

Illustration: L = 10M, RK 5.5%, RF taken as 5.0%

Rate difference	5.5% − 5.0%	**0.50%**
Net payment at year 2	10M × 0.50% × 1	**50,000**
Value, receiver of fixed	50,000 ÷ 1.04²	**46,228**
Value, payer of fixed	−50,000 ÷ 1.04²	**−46,228**
Value if RK = RF	0.00% × 10M × 1	**0**

IN THE LESSON

- Take a made up F R A on ten million dollars, for the year from year one to year two.
- The agreed fixed rate is five and a half per cent, while the forward rate is taken as five per cent.
- The difference is zero point five per cent.
- The net payment at year two is ten million times zero point five per cent times one year: fifty thousand dollars, received by the receiver of fixed.
- Discount it at the two year zero rate of four per cent, dividing by one point zero four squared, and the value today is forty six thousand, two hundred and twenty eight.
- The payer of fixed has the same number with the sign flipped.
- And if the fixed rate had equaled the forward rate, the value would be zero.

27 Bootstrapping a zero curve

- Build the zero curve from bond prices, short to long
- 6-month bill: zero = 2 × (100 ÷ P − 1), semiannual
- Later bonds: subtract the PV of earlier coupons first
- Remainder = final cash flow ÷ (1 + z)^n
- Zero = (final cash flow ÷ remainder)^(1/n) − 1

IN THE LESSON

- Where do spot rates come from in practice?
- From bootstrapping: building the zero curve from bond prices, one maturity at a time, from short to long.
- Short instruments such as bills have a single cash flow, so the zero rate follows directly.
- For a six month bill with price P and face one hundred, the semiannual zero rate is two times the quantity one hundred over P, minus one.
- For the next coupon bond, discount the earlier coupons with the zero rates already found.
- What is left of the price equals the final cash flow discounted at the one unknown zero rate.
- Solve it with an nth root, then move on to the next maturity.
- Never use yields to maturity in place of zero rates.

28 Example 11: bootstrapping two points

Illustration: 1y bill at 96; 2y bond, 5% coupon, price 98.50

1-year zero rate	100 ÷ 96 − 1	**4.17%**
PV of year-1 coupon	5 ÷ 1.0417	**4.80**
Left for the final flow	98.50 − 4.80	**93.70**
2-year zero rate	(105 ÷ 93.70)^½ − 1	**5.86%**

IN THE LESSON

- Here are made up prices.
- A one year bill with face one hundred costs ninety six.
- Its zero rate is one hundred over ninety six, minus one: four point one seven per cent.
- Next a two year bond with a five per cent annual coupon is priced at ninety eight fifty.
- The year one coupon of five is discounted at the zero rate we just found, which is five divided by one point zero four one seven, or four eighty.
- Take that away from the price and ninety three seventy is left to pay for the final cash flow of one hundred and five.
- The two year zero rate is the square root of one hundred and five over ninety three seventy, minus one: five point eight six per cent.

29 Four term structure theories

Why do yields differ by maturity?

Theory	Core idea
Pure expectations	Forwards = expected spots
Liquidity preference	Premium grows with maturity
Market segmentation	Separate supply and demand
Preferred habitat	Moves for more yield

IN THE LESSON

- Why do yields differ by maturity?
- Four theories compete.
- Pure expectations says a long rate is an average of expected future short rates.
- So forward rates equal expected spot rates, and an upward slope signals expected rate rises.
- Liquidity preference says investors prefer short maturities, so forward rates exceed expected spot rates by a premium that grows with maturity.
- That explains why curves usually slope upward even when expectations are flat.
- Market segmentation says each maturity range has its own supply and demand, with no substitution between ranges.
- Preferred habitat is the middle view: investors favor a range but will move for extra yield.
- Only pure expectations implies that the maturity mix of debt supply has no effect.

30 Example 12: expectations and premiums

Illustration: spot 3% (1y), 4% (2y); expected 1y rate next year 4.5%

Forward rate, year 1 to 2	$1.04^2 \div 1.03 - 1$	**5.01%**
Expected 1y rate next year	assumed	**4.50%**
Premium (liquidity pref.)	5.01% − 4.50%	**0.51%**
2y if pure expectations	√(1.03 × 1.045) − 1	**3.75%**
Actual 2y spot	given	**4.00%**

IN THE LESSON

- Test the theories with made up numbers.
- The one year spot rate is three per cent and the two year spot rate is four per cent, so the forward rate for the second year is five point zero one per cent.
- Suppose investors expect next year's one year rate to be four and a half per cent.
- The gap between the forward and the expected rate is zero point five one per cent, which liquidity preference would call the premium.
- Under pure expectations, there would be no premium, and the two year rate would be the square root of one point zero three times one point zero four five, minus one: three point seven five per cent.
- The actual two year rate of four per cent sits above it.

31 Four mistakes to avoid

- Mixed frequencies: convert to one frequency first
- One rate for all flows: use each cash flow's spot rate
- Macaulay for modified: divide by (1 + y/m) first
- Averaging spots: forwards use growth factor ratios

IN THE LESSON

- Here are four mistakes that come up again and again in this topic.
- First, comparing quotes at different compounding frequencies.
- Convert them to one frequency first, using the log and exponential conversions.
- Second, using one rate for every cash flow.
- Each coupon needs the spot rate for its own date, and one average yield is only a summary.
- Third, mixing up Macaulay and modified duration.
- Divide Macaulay by one plus the yield over the frequency before using it for price changes, and put yield changes in decimals.
- Fourth, averaging spot rates to find a forward rate.
- Forwards come from ratios of growth factors, with annual compounding, or from differences of rate times time, with continuous compounding.

32 Recap

- Treasury, repo, SOFR: SOFR is secured and overnight
- Compounding: A(1 + R/m)^(mT), or A·e^(RT) continuous
- Price = PV of flows, at the yield or each spot rate
- $\Delta P/P \approx -D \cdot \Delta y + \frac{1}{2} \cdot C \cdot (\Delta y)^2$, yield changes in decimals
- Forwards, FRAs, bootstrapping, four term theories

IN THE LESSON

- Let's pull it together.
- Treasury yields, repo rates and S O F R are the benchmark rates, and S O F R is secured, overnight and published by the New York Fed.
- Compounding more often raises terminal value by less each time, and quotes must be converted to one frequency before comparing.
- A bond is worth the present value of its cash flows, discounted at its yield or, more exactly, at each spot rate.
- Duration, dollar duration and the dollar value of a basis point measure sensitivity, and convexity corrects the straight line.
- Forward rates come from the spot curve, a forward rate agreement is valued from them, zero rates are bootstrapped from prices, and four theories explain the slope of the curve.

33 Try three quick questions

1. 4% compounded semiannually: continuous equivalent?
   Answer: $2 \times \ln(1.02) = 3.96\%$.
2. Price 90, modified duration 4, yield +50 bp. Price change?
   Answer: $-4 \times 0.005 = -2\%$, about −1.80.
3. Spots 3% (2y), 4% (3y), annual. Forward, year 2 to 3?
   Answer: $1.04^3 \div 1.03^2 - 1 = 6.03\%$.

IN THE LESSON

- Now try three quick questions yourself.
- Pause after each one.
- Question one.
- A rate of four per cent is compounded semiannually.
- What is the equivalent continuous rate?
- Question two.
- A bond has a price of ninety and a modified duration of four.
- Yields rise by fifty basis points.
- What is the approximate price change?
- Question three.
- The two year spot rate is three per cent and the three year spot rate is four per cent, both with annual compounding.
- What is the one year forward rate from year two to year three?
- The answers are on the board.
- Check each one, and if you slipped, go back to the matching example in this lesson.