CHARTA LEARNING · LESSON NOTES

Expectations, Risk Premium, Convexity, and the Shape of the Term Structure

FRM Part II

1 A story: three forces behind one curve

- Kestrel's curve: 1 year 3%, 2 years 3.5%, 3 years 4%
- Junior trader: the market expects rates to rise
- Force 1, expectations: where short rates are going
- Force 2, risk premium: extra pay for long bonds
- Force 3, convexity: uncertainty pushes long yields down

IN THE LESSON

- Kestrel's head of rates, Mira, looks at today's curve.
- One year money pays three per cent, two year money three and a half, three year money four.
- The curve slopes upward.
- A junior trader says the market must expect rates to rise.
- Mira says that is only one of three forces.
- First, expectations: where the market thinks short rates are going.
- Second, a risk premium: extra pay that investors demand for holding long bonds, whose prices swing more.
- Third, convexity: because bond prices curve, uncertainty about rates pushes long yields down.
- This lesson takes the three forces one at a time.

2 Spot rates and forward rates

- Spot rate z_n: yield today on an n-year zero-coupon bond
- Forward rate: a future period's rate locked in today
- Spot rate: geometric average of the one-year path
- Multiply growth factors and take a root; do not average
- Pure expectations: forward rate = expected short rate

IN THE LESSON

- First, two words.
- A spot rate is the yield today on a zero coupon bond, a bond that pays one sum at maturity and nothing before.
- Write z one, z two and so on for the one year, two year and three year spot rates.
- A forward rate is the rate for a future period that today's curve locks in, such as the rate from year one to year two.
- The key idea is that a spot rate is the geometric average of the one year rates along the path.
- That means you multiply the growth factors and take a root, not add the rates and divide.
- Under the pure expectations hypothesis, each forward rate equals the market's expected future short rate.

3 From spot rates to forward rates

Factor_2 = (1 + z_1) × (1 + f_2) = (1 + z_2)^2
Fwd = (1 + z_n)^n ÷ (1 + z_m)^m − 1

Factor	Growth factor over n years
z_n	Spot rate for n years
f_n, Fwd	Forward rate for year n
m	m = n − 1, one year less

IN THE LESSON

- Here are the formulas.
- Write Factor two for the two year growth factor.
- In terms of the spot rate, it is one plus z two, squared.
- Investing for two years at the spot rate must grow money as much as investing for one year at z one, then rolling over at the forward rate f two.
- So one plus z one, times one plus f two, equals one plus z two squared.
- In general, let m be n minus one, one year less than n.
- The forward rate for year n is one plus z n to the power n, divided by one plus z m to the power m, minus one.

4 Example 1: Kestrel's forward rates

Kestrel's curve: z_1 = 3%, z_2 = 3.5%, z_3 = 4%. Annual compounding

Two-year factor	1.035^2	**1.071225**
Forward rate, year 2	1.071225 ÷ 1.03 − 1	**4.00%**
Three-year factor	1.04^3	**1.124864**
Forward rate, year 3	1.124864 ÷ 1.071225 − 1	**5.01%**
Pure expectations path	years 1, 2, 3	**3%, 4.00%, 5.01%**

IN THE LESSON

- The two year growth factor is one point zero three five squared, which is one point zero seven one two two five.
- Divide by one point zero three, the one year factor, and subtract one.
- The forward rate for year two is four point zero zero per cent.
- The three year factor is one point zero four cubed, one point one two four eight six four.
- Divide by the two year factor, one point zero seven one two two five, and subtract one.
- The forward rate for year three is five point zero one per cent.
- So, under the pure expectations hypothesis, the market expects the one year rate to go from three per cent today, to about four per cent next year, and about five per cent the year after.

5 Example 2: an inverted curve

One year later: z_1 = 6%, z_2 = 5%. The curve has inverted

Two-year factor	1.05^2	**1.1025**
Forward rate, year 2	1.1025 ÷ 1.06 − 1	**4.01%**
Reading	expected to fall	**6% to 4.01%**
Shortcut: average the rates	2 × 5% − 6%	**4.00%: not exact**

IN THE LESSON

- Now suppose, one year later, Kestrel's curve has inverted, with long rates below short rates.
- The one year rate is six per cent and the two year rate is five per cent.
- The two year factor is one point zero five squared, one point one zero two five.
- Divide by one point zero six and subtract one, and the forward rate for year two is four point zero one per cent.
- Under pure expectations, the market expects the one year rate to fall from six per cent to about four per cent.
- A shortcut that averages rates, twice five minus six, gives four point zero zero per cent.
- Close, but not exact.

6 Reading the shape of the curve

- Upward slope: forwards above spot, rates expected to rise
- Inverted: forwards below spot, rates expected to fall
- Flat: no change expected
- Compound consistently with the quote
- Never average rates arithmetically

IN THE LESSON

- What do the shapes mean under pure expectations?
- An upward sloping curve means the forwards sit above the spot rates, so short rates are expected to rise.
- An inverted curve means forwards sit below spot rates, so short rates are expected to fall.
- A flat curve means no change is expected.
- Two working rules keep the arithmetic honest.
- Compound consistently with the way the rates are quoted: annual rates with annual factors, continuous rates with exponentials.
- And never average rates arithmetically.
- A spot rate is a geometric average of the path.
- The averaging shortcut is only an approximation, and the exam may test the exact answer.

7 Adding a risk premium

Forward = Expected + Premium
Expected = Forward − Premium

Forward	Rate implied by today's curve
Expected	Market's expected short rate
Premium	Extra pay for bearing rate risk

IN THE LESSON

- Now the second force.
- Investors who lock money up for longer face bigger price swings, so they may demand extra yield, a liquidity or term premium.
- If they do, the forward rate implied by the curve includes it.
- Forward equals Expected plus Premium.
- Turn that around.
- The expected rate is the forward rate less the premium.
- So the forward is no longer a clean forecast.
- It overstates the expected rate by the size of the premium.
- The exam point is that, under the pure form of the hypothesis, the forward equals the expected rate, and any premium must be stripped out explicitly before you read the forward as a forecast.

8 Example 3: stripping out the premium

Forwards 4.00% (year 2) and 5.01% (year 3). Premia 0.40% and 0.80%

Expected rate, year 2	4.00% − 0.40%	**3.60%**
Expected rate, year 3	5.01% − 0.80%	**4.21%**
Expected path	years 1, 2, 3	**3%, 3.60%, 4.21%**

Rates still expected to rise, but less than the raw forwards say

IN THE LESSON

- Mira's view is that investors demand a term premium of point four zero per cent for the second year and point eight zero per cent for the third.
- So she strips the premium out of each forward.
- For year two, the forward of four point zero zero per cent, less point four zero, gives an expected rate of three point six zero per cent.
- For year three, the forward of five point zero one per cent, less point eight zero, gives an expected rate of four point two one per cent.
- The expected path is three, then three point six zero, then four point two one per cent.
- Rates are still expected to rise, but by much less than the raw forwards suggested.

9 Example 4: a rising curve with no expected rise

Expected one-year rate stays at 3%. Premium for year 2 is 1%

Forward, year 2	3% + 1%	**4%**
Two-year spot rate	√(1.03 × 1.04) − 1	**3.50%**
Shape of the curve	3% then 3.50%	**upward**
Expected change in rates	3% to 3%	**none**

IN THE LESSON

- Here is the trap.
- Suppose everyone expects the one year rate to stay at three per cent next year.
- But investors demand a premium of one per cent for the second year.
- The forward rate for year two is the expected rate plus the premium, three plus one, which is four per cent.
- The two year spot rate is the geometric average of three and four per cent: the square root of one point zero three times one point zero four, minus one, which is three point five zero per cent.
- So the curve slopes upward, from three to three and a half per cent, although nobody expects rates to rise.
- An upward slope alone does not prove that rates are expected to go up.

10 A risk-neutral rate tree

- Lattice of possible one-year rates; up or down at each node
- Branch probabilities set by the model
- Arbitrage-free and calibrated to market prices
- Some trees use ½ up and ½ down
- Backward induction from the face value at maturity

IN THE LESSON

- Now the third force, volatility, and the tool to see it: a risk neutral rate tree.
- A rate tree is a lattice of possible one year rates.
- From each node, the rate can step up or down.
- Each branch carries a probability set by the model, chosen so that the tree is free of arbitrage and is calibrated to market prices.
- Some trees use a convention of one half up and one half down.
- A bond is valued by backward induction.
- You start at maturity with the face value, then step back one node at a time, taking the probability weighted value and discounting at that node's rate.

11 One step back in the tree

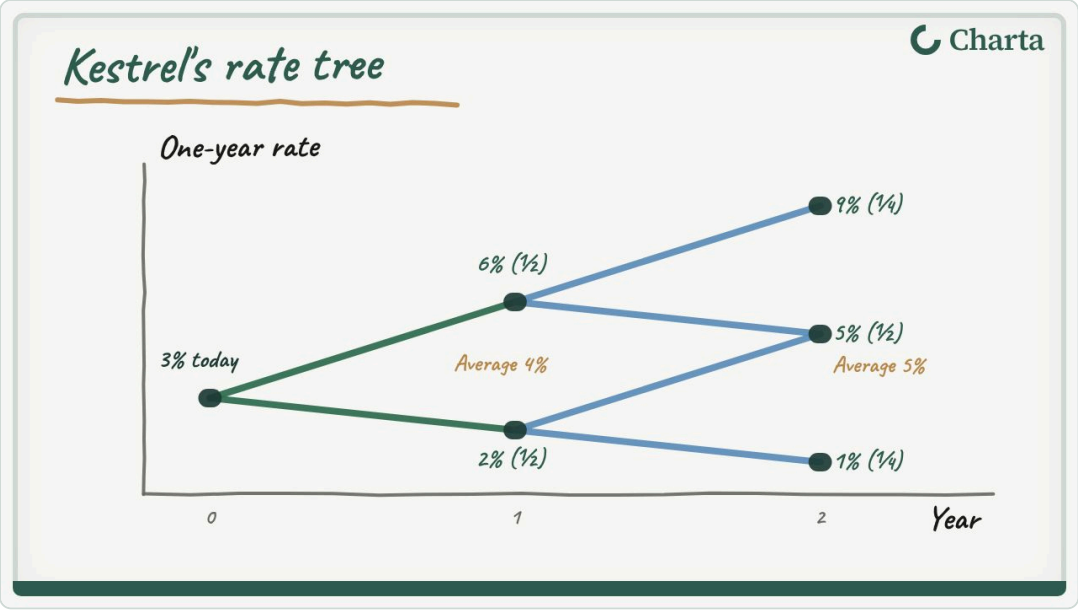
Value = (0.5 × V_up + 0.5 × V_down) ÷ (1 + r)

Value	Value at this node
V_up	Value one step later, rate up
V_down	Value one step later, rate down
r	One-year rate at this node

IN THE LESSON

- Here is one step of backward induction.
- Let V up and V down be the bond's values one step later, after the rate has moved up or down.
- Let r be the one year rate at the node where you stand.
- With branch probabilities of one half, the value at the node is point five times V up, plus point five times V down, all divided by one plus r.
- Two details matter.
- You average the values, not the rates.
- And you discount by the rate at the node where you stand, not by the rate at the next nodes.
- Repeat back to today, and the first node's value is the price.

12 Kestrel's rate tree



IN THE LESSON

- Here is Kestrel's tree.
- Along the bottom is time in years, and up the side is the one year rate.
- For illustration, Mira centres the tree on the forwards, rounded to three, four and five per cent, not on her own expected path, and adds volatility.
- These are risk neutral rates, with no premium.
- The first branches go from today to year one, and the next branches to year two.
- Today the rate is three per cent.
- In one year it is either six per cent or two per cent, each with probability one half, so the average is four.
- In two years it is nine, five or one per cent.
- Nine needs two up moves, a chance of one quarter.
- Five can be reached by up then down, or down then up, so its chance is one half.
- One per cent has a chance of one quarter.
- The average is five per cent.

13 Example 5: a two-year zero in the tree

Face value $100, paid at the end of year 2. Branch probabilities ½

Year 1, rates 6% and 2%	100 ÷ 1.06 and 100 ÷ 1.02	**94.34, 98.04**
Average of the two	(94.34 + 98.04) ÷ 2	**96.19**
Price today	96.19 ÷ 1.03	**93.39**
Spot yield z_2	√(100 ÷ 93.39) − 1	**3.48%**
Market price, z_2 = 3.5%	100 ÷ 1.035^2	**93.35**

Calibrated tree: year-1 rates about 6.04% and 2.04% give 93.35

IN THE LESSON

- Price a two year zero paying one hundred dollars at the end of year two.
- Step back to year one.
- At the six per cent node it is worth one hundred over one point zero six, and at the two per cent node one hundred over one point zero two: ninety four point three four and ninety eight point zero four.
- Back to today.
- Average the two, ninety six point one nine, and discount at three per cent.
- The price is ninety three point three nine.
- Its spot yield, the square root of one hundred over ninety three point three nine, minus one, is three point four eight per cent.
- But the market's price, one hundred over one point zero three five squared, is ninety three point three five.
- This illustrative tree does not reproduce it.
- A tree used for pricing must, so it is calibrated: year one rates of about six point zero four and two point zero four per cent give ninety three point three five.

14 Example 6: a three-year zero in the tree

Face value $100, paid at the end of year 3. Year-2 rates 9%, 5%, 1%

Year 2 values	100 ÷ 1.09, 1.05, 1.01	**91.74/95.24/99.01**
Year 1, rate 6%	(91.74 + 95.24) ÷ 2 ÷ 1.06	**88.20**
Year 1, rate 2%	(95.24 + 99.01) ÷ 2 ÷ 1.02	**95.22**
Price today	(88.20 + 95.22) ÷ 2 ÷ 1.03	**89.04**
Spot yield z_3	(100 ÷ 89.04)^(1/3) − 1	**3.95%**

No volatility: 100 ÷ (1.03 × 1.04 × 1.05) = 88.91, a yield of 4.00%

IN THE LESSON

- Now a three year zero.
- At year two, one year from maturity, it is worth one hundred divided by one point zero nine, one point zero five and one point zero one: ninety one point seven four, ninety five point two four and ninety nine point zero one.
- Step back to the six per cent node.
- Average the top two values and divide by one point zero six, to get eighty eight point two zero.
- At the two per cent node, average the bottom two and divide by one point zero two, ninety five point two two.
- Back to today: average those two and divide by one point zero three.
- The price is eighty nine point zero four, and the spot yield is three point nine five per cent.
- Without volatility, one hundred divided by one point zero three, times one point zero four, times one point zero five, is eighty eight point nine one, a yield of four point zero zero per cent.

15 More volatility, lower yields

Same average path 3%, 4%, 5%; only the spread of rates changes

Tree	2-year yield	3-year yield
No volatility	3.50%	4.00%
Steps of 1 point	3.49%	3.98%
Steps of 2 points	3.48%	3.95%
Gap, 2-point tree	about 2 bp	about 5 bp

IN THE LESSON

- Now Mira compares three trees with the same average path of three, four and five per cent.
- With no volatility, the two year yield is three point five zero per cent and the three year yield is four point zero zero.
- With steps of one percentage point up or down, so year one rates of five and three, the yields are three point four nine and three point nine eight.
- With steps of two points, our tree, they are three point four eight and three point nine five.
- Measured from the unrounded figures, the wide tree's gap is about two basis points at two years, and about five at three years.
- More volatility means lower yields, and the effect grows with maturity.

16 Why volatility pulls the curve down

- Bond price is a convex function of the rate
- Rates down: price rises more than it falls for rates up
- Wider spread, same average rate: higher expected price
- Higher price, lower spot yield: curve below expected rates
- Effect grows with maturity

IN THE LESSON

- Why does this happen?
- A bond's price is a convex function of the rate: the price curve bends upward.
- When rates fall, the price rises by more than it falls when rates rise by the same amount.
- So if you spread the possible rates wider while keeping the same average rate, the average price goes up.
- A higher price means a lower spot yield.
- That is why volatility pulls the curve below the path of expected rates.
- And the effect grows with maturity, because longer bonds have more convexity and face more years of rate uncertainty.
- On a long bond, it can outweigh a modest risk premium.

17 Three mistakes with a rate tree

Two-year zero: correct price 93.39 (from Example 5)

Discount at the average rate	100 ÷ (1.03 × 1.04)	**93.35: too low**
Skip the first discount	stop at the average	**96.19: too high**
Weight nodes equally	not ⅓ each: count paths	**¼, ½, ¼**

IN THE LESSON

- The first mistake is discounting at the average rate.
- For the two year zero, dividing one hundred by one point zero three times one point zero four gives ninety three point three five.
- That is the price with no volatility, so it is too low, and misses the convexity effect entirely.
- The second is skipping the discount at the first node.
- Stopping at the average of ninety six point one nine is too high, because it forgets that today's rate of three per cent still has to be applied.
- The third is weighting the nodes of a recombining tree equally, where up then down lands where down then up does.
- The year two nodes do not get one third each.
- Count the paths: their probabilities are one quarter, one half and one quarter.

18 Jensen's inequality

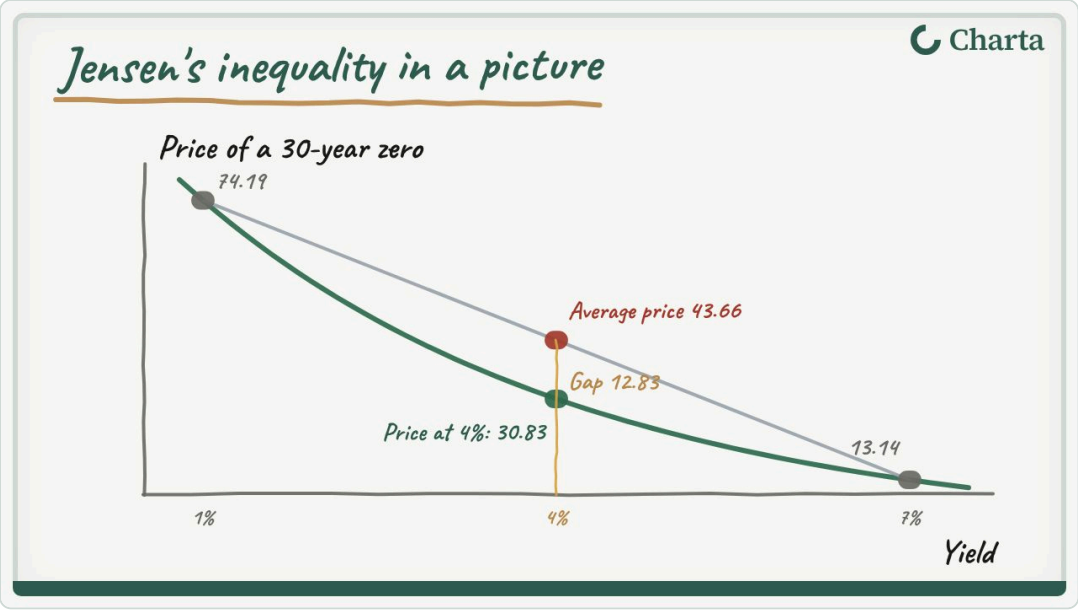
E[f(X)] ≥ f(E[X])
Gap grows with the dispersion of X

f	A convex function, e.g. bond price
X	A random variable, e.g. the rate
E[ ]	Expected value: the average

IN THE LESSON

- Jensen's inequality explains the tree's result.
- For a convex function f, the average of f is at least f of the average.
- In symbols, E of f of X is greater than or equal to f of E of X.
- Here X is a random variable, like next year's rate, and E means the expected value, the probability weighted average.
- The gap between the two sides grows with the dispersion of X, how widely it is spread.
- A bond price is a convex function of the rate.
- So the expected price, across uncertain future rates, is higher than the price at the expected rate.
- That is the convexity effect, and it is why the yield implied by a zero coupon price is below the expected average rate.

19 Jensen's inequality in a picture



IN THE LESSON

- Here is the picture.
- To make the bend easy to see, take a thirty year zero and two widely spread yields.
- Along the bottom is the yield.
- Up the side is the price per one hundred dollars of face value.
- The green curve is the price.
- It bends upward, so it is convex.
- The grey straight line joins two equally likely outcomes, a yield of one per cent and a yield of seven per cent.
- The dotted guide marks the average yield, four per cent.
- At one per cent the price is seventy four point one nine, and at seven per cent it is thirteen point one four.
- The average of the two prices, forty three point six six, sits on the line.
- The price at the average yield, thirty point eight three, sits on the curve, well below.
- The gap is twelve point eight three.
- A shorter bond or closer yields shrink it, but it is always there.

20 Example 7: Jensen's gain, exactly

10-year zero, face $100. Yield 3% or 5%, equally likely

Price at 3%	100 ÷ 1.03^10	**74.41**
Price at 5%	100 ÷ 1.05^10	**61.39**
Expected price	(74.41 + 61.39) ÷ 2	**67.90**
Price at expected yield 4%	100 ÷ 1.04^10	**67.56**
Convexity gain	67.90 − 67.56	**0.34**

Yield implied by 67.90: (100 ÷ 67.90)^(1/10) − 1 = 3.95%, below 4%

IN THE LESSON

- Kestrel holds a ten year zero with a face value of one hundred dollars.
- Suppose its yield is about to jump to three per cent or five per cent, with equal chances, so the expected yield is four per cent.
- At three per cent the price is one hundred over one point zero three to the power ten, seventy four point four one.
- At five per cent it is sixty one point three nine.
- The expected price is the average, sixty seven point nine zero.
- The price at the expected yield of four per cent is only sixty seven point five six.
- The gap, point three four, is the convexity gain.
- The yield implied by the expected price is three point nine five per cent, below the expected yield of four.

21 Estimating the convexity effect

Gain ≈ ½ × C × σ^2 × P
C ≈ T × (T + 1) ÷ (1 + y)^2
Fall in yield ≈ ½ × σ^2 × T

C	Convexity of the bond
σ	Yield volatility; σ² is variance
P	Bond price
T	Maturity in years

IN THE LESSON

- In practice we estimate the effect with a formula.
- Let C be the bond's convexity, sigma the volatility of its yield, and P the bond's price.
- The gain in expected price is about one half, times C, times sigma squared, times P.
- Sigma squared is the variance.
- For a zero coupon bond with annual compounding, the convexity is roughly T times T plus one, divided by one plus y, squared.
- Here T is the maturity and y the yield.
- So the effect rises with variance, and roughly with the square of maturity.
- With continuous compounding and a random parallel shift in yields, the yield falls by about one half, times sigma squared, times T.
- These are approximations.

22 Example 8: the approximation checks out

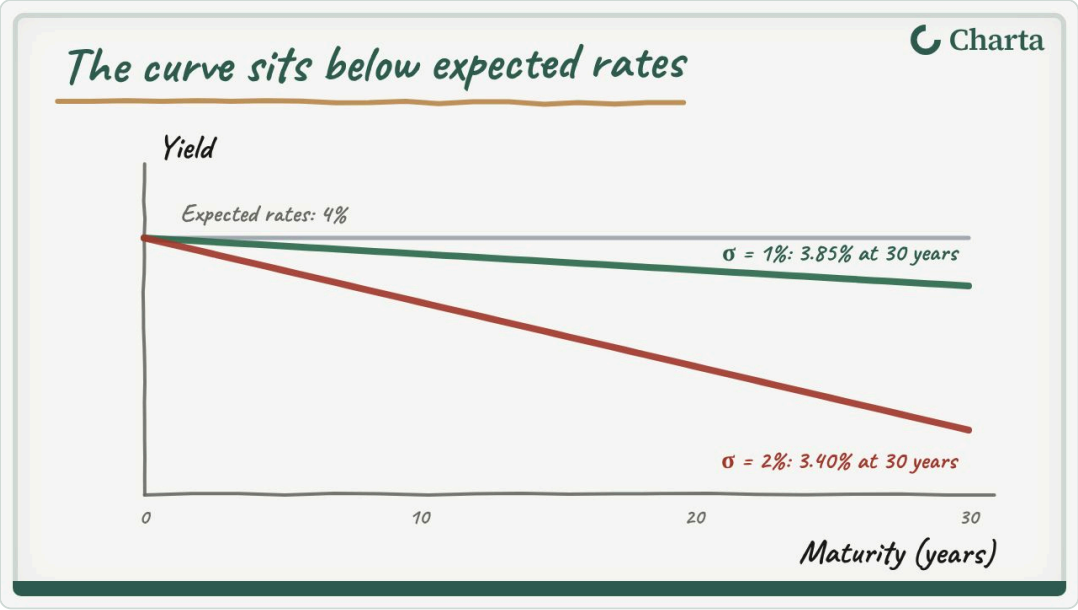
10-year zero at 4%. Yield 3% or 5%, so σ = 1% and σ² = 0.0001

Convexity C	10 × 11 ÷ 1.04^2	**101.70**
Gain in expected price	½ × 101.70 × 0.0001 × 67.56	**0.34**
Exact gain, Example 7	67.90 − 67.56	**0.34**
Fall in yield	½ × 0.0001 × 10	**0.05% = 5 bp**
Exact fall	4% to 3.95%	**about 5 bp**

IN THE LESSON

- On the ten year zero, the two yields, three and five per cent, sit one point either side of four, so sigma is one per cent and the variance is point zero zero zero one.
- Convexity at a four per cent yield is ten times eleven, one hundred and ten, divided by one point zero four squared, one point zero eight one six.
- That is one hundred and one point seven zero.
- The gain is one half times one hundred and one point seven zero, times point zero zero zero one, times sixty seven point five six, which is point three four.
- That matches.
- The yield falls by about one half times point zero zero zero one times ten, point zero five per cent, or five basis points.
- The exact fall, four per cent to three point nine five, agrees.

23 The curve sits below expected rates



IN THE LESSON

- Here is what that does to the whole curve.
- Along the bottom is maturity, out to thirty years.
- Up the side is the yield.
- Suppose expected rates are four per cent at every maturity: the grey line.
- Now let the whole curve face one random parallel shift.
- Here sigma is the volatility of yields under that single shift, not the volatility of the short rate.
- The green line uses a sigma of one per cent.
- Each spot yield falls by about one half of point zero zero zero one times T, five basis points for every ten years, so at thirty years it is three point eight five per cent.
- The red line uses two per cent.
- Doubling sigma quadruples the variance, so the curve falls four times as fast, to three point four zero per cent.
- This is only an approximation.
- If the short rate itself moves randomly each year, the effect can grow faster with maturity.

24 Example 9: maturity and variance

Same 4% yield and σ = 1%. Gain as a share of price = ½ × C × σ²

Convexity, 20-year zero	20 × 21 ÷ 1.04^2	**388.31**
Ratio to the 10-year	388.31 ÷ 101.70	**3.82 times**
Gain, 10-year	½ × 101.70 × 0.0001	**0.51%**
Gain, 20-year	½ × 388.31 × 0.0001	**1.94%**
Gain, 10-year, σ = 2%	½ × 101.70 × 0.0004	**2.03%**

IN THE LESSON

- First, maturity.
- The ten year zero's convexity is one hundred and one point seven zero.
- A twenty year zero at the same four per cent yield has convexity twenty times twenty one, four hundred and twenty, over one point zero eight one six, which is three hundred and eighty eight point three one.
- That is three point eight two times as much, close to the square of two, which is four.
- As a share of price, the gain is one half times C times sigma squared: point five one per cent for the ten year, and one point nine four per cent for the twenty year.
- Second, variance.
- Double the volatility to two per cent and the ten year's gain quadruples, to two point zero three per cent.

25 Mistakes with Jensen's inequality

10-year zero: correct gain 0.34 (from Examples 7 and 8)

Reverse the inequality	67.90 vs price at 4%, 67.56	**67.90 is larger**
Omit the ½	101.70 × 0.0001 × 67.56	**0.69: double**
Volatility for variance	½ × 101.70 × 0.01 × 67.56	**34.35: absurd**

IN THE LESSON

- Three mistakes cost marks here.
- The first is reversing the inequality, and saying the price at the expected rate is higher than the expected price.
- For a convex price, it is the other way round: sixty seven point nine zero beats sixty seven point five six.
- The second is omitting the one half.
- For the ten year zero, that doubles the gain, from about point three four to about point six nine.
- The third is using volatility where variance belongs.
- Using point zero one instead of point zero zero zero one gives a gain of thirty four point three five dollars, half of the bond's price, which is absurd.

26 The four pieces of a bond's return

- Carry: about the short rate
- Rate changes: rolldown, or moves against forwards
- Risk premium: pay for bearing rate risk
- Convexity effect: positive, by Jensen's inequality
- Risk-averse investors are paid, so longer bonds earn more

IN THE LESSON

- Now the last question: what can an investor expect to earn from holding a bond for a short horizon, say a year?
- The expected return has four pieces.
- First, the short rate carry, the return from simply holding it, which over a short period is about the short rate.
- Second, the price effect of expected rate changes.
- If the curve stays unchanged, this is rolldown, the gain as the bond ages into a lower yield.
- Otherwise it is the move in rates relative to the forwards.
- Third, a risk premium, pay for bearing interest rate risk.
- Fourth, a convexity effect, which is positive by Jensen's inequality.
- A risk averse investor is paid for rate risk, so longer bonds earn more.

27 The expected return formula

Premium = $D \times \lambda \times \sigma$
Return = $r + D \times \lambda \times \sigma + Conv - D \times \Delta r$

r	Short rate: carry
D	Duration of the bond
λ	Market price of rate risk
Conv	Convexity effect, ½ × C × σ^2
Δr	Expected change in rates

IN THE LESSON

- Here are the pieces as a formula.
- Let r be the short rate, the carry.
- Let D be the bond's duration, and lambda the market price of risk: the extra return investors demand for each unit of rate volatility, sigma, that they bear.
- For a bond of duration D, the expected excess return over the short rate is the premium: D times lambda times sigma.
- Let Conv be the convexity effect, one half times C times sigma squared, and delta r the expected change in rates.
- Adding the four pieces, the return is r, plus D times lambda times sigma, plus Conv, minus D times delta r.
- Risk neutral pricing sets the premium to zero.
- It is a valuation device, not a forecast of what a holder earns.

28 Example 10: Kestrel's ten-year bond

D = 8, C = 80. Short rate 3%, λ = 0.25, σ = 1%, Δr = +0.10%

1. Carry: short rate		**3.00%**
2. Risk premium	8 × 0.25 × 1%	**2.00%**
3. Convexity effect	½ × 80 × 0.0001	**0.40%**
4. Expected rate rise	− 8 × 0.10%	**−0.80%**
Expected return	3.00 + 2.00 + 0.40 − 0.80	**4.60%**

IN THE LESSON

- Kestrel holds a ten year bond with a duration of eight and a convexity of eighty.
- The short rate is three per cent.
- The market price of risk is point two five, rate volatility is one per cent, and the desk expects rates to rise by point one zero per cent over the year.
- First, the carry: three per cent.
- Second, the risk premium: eight times point two five times one per cent, two point zero zero per cent.
- Third, the convexity effect: one half times eighty times point zero zero zero one, point four zero per cent.
- Fourth, the expected rate rise: eight times point one zero, a loss of point eight zero per cent.
- Add them up.
- The expected return is four point six zero per cent.

29 Example 11: short bonds, and the risk-neutral view

2-year bond: D = 2, C = 6. Same r, λ, σ and Δr as Example 10

Risk premium	2 × 0.25 × 1%	**0.50%**
Convexity effect	½ × 6 × 0.0001	**0.03%**
Expected rate rise	− 2 × 0.10%	**−0.20%**
Expected return, 2-year	3.00 + 0.50 + 0.03 − 0.20	**3.33%**
10-year, risk-neutral, λ = 0	3.00 + 0.40 − 0.80	**2.60%**

Risk-neutral 2.60% is a valuation figure, not a forecast

IN THE LESSON

- Now a two year bond with a duration of two and a convexity of six.
- Its risk premium is two times point two five times one per cent, point five zero per cent.
- Its convexity effect is one half times six times point zero zero zero one, point zero three per cent.
- The expected rate rise costs two times point one zero, point two zero per cent.
- So its expected return is three plus point five zero, plus point zero three, minus point two zero: three point three three per cent, well below the ten year's four point six zero.
- Longer bonds earn more because they carry more rate risk.
- Under risk neutral pricing, lambda is zero, and the ten year's figure would be three plus point four zero minus point eight zero, two point six zero per cent.
- That two point six zero is a valuation figure, not a forecast of what Kestrel will earn.

30 Example 12: rolldown, and why yield is not return

5-year zero at a 4% yield. Curve unchanged; 4-year yield is 3.7%

Price today	100 ÷ 1.04^5	**82.19**
Price in one year	100 ÷ 1.037^4	**86.47**
One-year return	86.47 ÷ 82.19 − 1	**5.21%**
Rolldown on top of the yield	5.21% − 4.00%	**1.21%**

IN THE LESSON

- Now rolldown.
- Kestrel holds a five year zero priced at a four per cent yield, so it costs one hundred over one point zero four to the power five, eighty two point one nine.
- Suppose the curve stays exactly as it is.
- In a year the bond is a four year zero, and the four year yield on today's curve is three point seven per cent.
- Its price is then one hundred over one point zero three seven to the power four, eighty six point four seven.
- The one year return is eighty six point four seven over eighty two point one nine, minus one, which is five point two one per cent.
- The yield was four per cent.
- Rolldown added one point two one per cent.
- A bond's yield is not its expected return.

31 Mistakes to avoid

- Dropping duration: premium 8 × 0.25% = 2%, not 0.25%
- Forgetting the short rate: the premium is an excess return
- Treating the yield as the expected return
- Risk-neutral pricing values bonds; it does not forecast

IN THE LESSON

- Three errors in expected return questions come up most often.
- The first is dropping duration from the premium.
- For the ten year bond, the premium is eight times point two five, two per cent, not lambda times sigma alone, point two five per cent.
- The second is forgetting the short rate.
- The premium is an excess return over the short rate, so the carry of three per cent must be added back.
- The third is treating the yield as the expected return.
- As we just saw, rolldown, rate changes, the premium and convexity all move the return away from the yield.
- And remember that risk neutral pricing values a bond.
- It does not forecast what a holder earns.

32 Try three quick questions

1. z_1 = 2%, z_2 = 3%. Year-2 premium 0.5%. Expected rate?
   Answer: Fwd = 1.03^2 ÷ 1.02 − 1 = 4.01%. Less 0.5%: 3.51%.
2. Convexity 50, rate volatility 2%. Gain as a share of price?
   Answer: ½ × 50 × 0.0004 = 1%.
3. D = 5, r = 2%, λ = 0.3, σ = 1%, Δr = 0, no convexity. Return?
   Answer: 2% + 5 × 0.3 × 1% = 3.5%.

IN THE LESSON

- Now three quick questions.
- Pause, think, then listen.
- Question one.
- The one year spot rate is two per cent and the two year spot rate is three per cent.
- Investors demand a term premium of point five per cent for the second year.
- What is the expected one year rate in a year?
- The forward is one point zero three squared over one point zero two, minus one, four point zero one per cent.
- Less the premium, the expected rate is three point five one per cent.
- Question two.
- A bond has convexity of fifty, and rate volatility is two per cent.
- What is the convexity gain, as a share of price?
- One half times fifty times point zero zero zero four, which is one per cent.
- Question three.
- Duration is five, the short rate two per cent, lambda point three and sigma one per cent, with no rate change and no convexity.
- What is the expected return?
- Two plus five times point three times one per cent, which is three point five per cent.