CHARTA LEARNING · LESSON NOTES

Arbitrage Pricing with Term Structure Models

FRM Part II

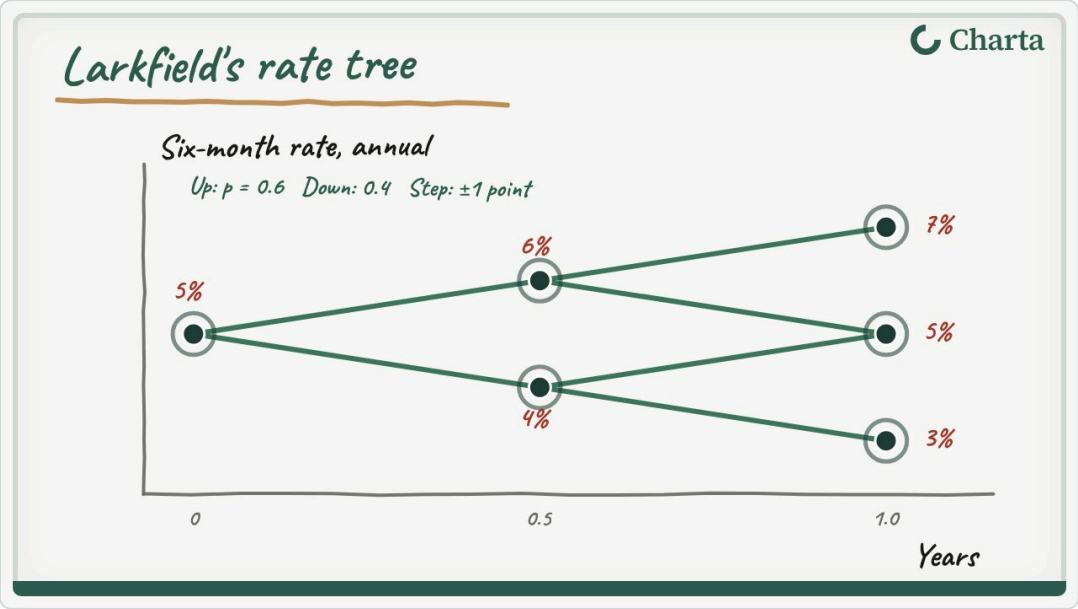
1 A story: the desk that used one rate

- Price a bond that pays $100 in one year
- Junior trader: today's rate for the whole year
- But the six-month rate in six months is unknown
- Binomial tree: the short rate moves up or down each step
- Each node has its own rate; work backward to today

IN THE LESSON

- Larkfield Bank's rates desk must price a bond that pays one hundred dollars in one year.
- A junior trader discounts it at today's six month rate for the whole year and calls it done.
- But the six month rate in six months is unknown; it may rise or fall.
- The desk's answer is a binomial tree: a map of the short rate, the rate on the shortest loan in the model, here six months, which at each step can only move up or down.
- Each node has its own rate, and we work backward from the bond's face value to today.

2 Larkfield's rate tree



IN THE LESSON

- Here is the tree the desk uses throughout.
- Across the bottom are the years; up the side is the six month rate, quoted as an annual rate with semiannual compounding.
- Today the rate is five per cent.
- In six months it is six per cent after an up move, or four after a down move.
- In one year it is seven, five or three per cent.
- Up always means the higher rate, and each move is one percentage point.
- The model gives an up move a probability of point six, and a down move point four, at every node.
- The dates are zero, half a year and one year, and each dot is a node.

3 Backward induction, one node at a time

Value = [p × Vup + (1 − p) × Vdown] ÷ (1 + r/2)

Value at this node	Worth at the node we stand on
Values at the next date	Vup and Vdown, one step later
Probability of an up move, p	Given by the model
This node's own annual rate	Halved for a six-month step

IN THE LESSON

- At every node the rule is the same.
- Value equals p times V up plus one minus p times V down, all divided by one plus r over two.
- Value is what the bond is worth at the node where we stand.
- V up and V down are its values one step later, after an up and a down move.
- p is the probability the model gives an up move.
- And r is this node's own rate.
- It is an annual rate, so for a six month step we halve it: six per cent becomes three.
- Start from the face value and work back, date by date.
- The value at the root is the price.

4 Example 1: the one-year zero

A zero pays $100 in one year. Rates: 5% today; 6% or 4% in six months. p = 0.6

Up node, rate 6%	100 ÷ 1.03	**97.0874**
Down node, rate 4%	100 ÷ 1.02	**98.0392**
Weighted average	0.6×97.0874 + 0.4×98.0392	**97.4681**
Root, rate 5%	97.4681 ÷ 1.025	**95.0908**

Price today: $95.0908 per $100 of face value

IN THE LESSON

- A zero coupon bond pays a single amount at maturity, here one hundred dollars in one year, so every node at one year is worth one hundred.
- At the six month up node the rate is six per cent, halved to three: one hundred divided by one point zero three is ninety seven point zero eight seven four.
- At the down node, four per cent halves to two: ninety eight point zero three nine two.
- At the root, point six times the up value plus point four times the down value is ninety seven point four six eight one.
- Divide by one point zero two five, the root's five per cent halved, to get ninety five point zero nine zero eight.
- That is the price per one hundred dollars of face value.

5 Example 2: three ways to get it wrong

Tree price 95.0908. Expected rate at six months: 0.6×6% + 0.4×4% = 5.2%

One rate: today's 5%	100 ÷ 1.025^2	**95.1814**
Expected rate 5.2% for 2nd half	100 ÷ (1.025 × 1.026)	**95.0887**
Annual rates not halved	(0.6×94.3396+0.4×96.1538)÷1.05	**90.5384**
Tree: each node's own rate	as in Example 1	**95.0908**

E[discount factor] ≠ discount factor at E[rate]

IN THE LESSON

- The tree price is ninety five point zero nine zero eight; the expected rate in six months is point six times six per cent plus point four times four per cent, five point two per cent.
- Now three errors.
- First, discounting the whole year at today's five per cent gives ninety five point one eight one four, too high.
- Using the expected rate for the second half gives ninety five point zero eight eight seven, close but wrong.
- Forgetting to halve gives six month values of ninety four point three three nine six and ninety six point one five three eight, and ninety point five three eight four today, far too low.
- Only each node's own rate gives ninety five point zero nine zero eight, because the expected value of a discount factor is not the discount factor at the expected rate.

6 Copying an option with the bond and cash

Delta = (Cup − Cdown) ÷ (Pup − Pdown)
Cash = (Cup − Delta × Pup) ÷ (1 + r/2)
Price = Delta × P0 + Cash

Option payoffs, Cup and Cdown	Option payoffs after the move
Bond prices, Pup and Pdown	Bond prices after the move
Bond's market price today, P0	Market price of the bond now
Delta: bonds held	Number of bonds in the copy

IN THE LESSON

- A client asks Larkfield for a call on the one year zero, expiring in six months.
- The desk prices it by arbitrage: it builds a portfolio of the bond and cash with the same payoff as the option in both the up and the down state.
- Same payoffs must mean the same price, or there is a riskless profit.
- Delta equals C up minus C down, over P up minus P down.
- Cash, usually a borrowing, equals C up minus Delta times P up, over one plus r over two.
- The price is Delta times P zero, plus Cash.
- C up and C down are the option's payoffs, P up and P down the bond's prices, P zero its price today, and Delta the bonds held.

7 Example 3: copying the call

Strike 97.50. Bond: 95.0908 today; 97.0874 if rates rise, 98.0392 if they fall

Payoff if rates rise	97.0874 is below 97.50	**0**
Payoff if rates fall	98.0392 − 97.50	**0.5392**
Delta	−0.5392 ÷ −0.9518	**0.5665**
Cash, a borrowing	−0.5665 × 97.0874 ÷ 1.025	**−53.6585**
Option price	0.5665 × 95.0908 − 53.6585	**0.2104**

IN THE LESSON

- The premise gives the strike, ninety seven point five zero, and the bond's prices: ninety five point zero nine zero eight today, then ninety seven point zero eight seven four if rates rise, or ninety eight point zero three nine two if they fall.
- If rates rise the bond is below the strike, so the call pays zero.
- If they fall it pays point five three nine two.
- Delta is minus point five three nine two over minus point nine five one eight: point five six six five bonds.
- Cash is minus point five six six five times ninety seven point zero eight seven four, over one point zero two five: a borrowing of fifty three point six five eight five.
- The price is point five six six five times ninety five point zero nine zero eight, minus that borrowing: point two one zero four dollars.

8 Checking the copy, and the classic mistake

Hold 0.5665 bonds; borrow 53.6585 for six months at 5% ÷ 2 = 2.5%

Repay the loan	53.6585 × 1.025	**55.0000**
Rates rise: portfolio	0.5665 × 97.0874 − 55	**0.0000**
Rates fall: portfolio	0.5665 × 98.0392 − 55	**0.5392**
Wrong: trader's own 70% fall	0.7 × 0.5392 ÷ 1.025	**0.3682**

The price uses traded prices only, never anyone's own probabilities

IN THE LESSON

- Check it.
- The desk holds point five six six five bonds and borrows fifty three point six five eight five for six months at two and a half per cent, half of five.
- It must repay fifty three point six five eight five times one point zero two five, which is fifty five.
- If rates rise, the bonds are worth exactly fifty five, so the portfolio is worth zero, like the call.
- If rates fall, the bonds are worth fifty five point five three nine two, leaving point five three nine two, again like the call.
- Now the classic mistake.
- A trader sure that rates will fall gives that a seventy per cent chance, and prices the call at point three six eight two.
- The price uses traded prices only, never anyone's own probabilities.

9 Risk-neutral pricing

Value = [p × Cup + (1 − p) × Cdown] ÷ (1 + r/2)
p = (Pdown − P0 × (1 + r/2)) ÷ (Pdown − Pup)

Probability of an up move, p	Risk-neutral chance of up
Bond's market price today, P0	Market price of the bond now
Bond prices, Pup and Pdown	Bond prices after the move
This node's own annual rate	Short rate, halved per step

IN THE LESSON

- Replication leads to a shortcut.
- Risk neutral pricing says a derivative is worth its expected payoff, computed with special probabilities and discounted at the riskless short rate.
- The first line: value equals p times C up plus one minus p times C down, over one plus r over two.
- The second line finds p from one bond: p equals P down minus P zero times one plus r over two, all over P down minus P up.
- Here p is the risk neutral chance of the up state, the higher rate.
- P zero is the bond's price today, P up and P down its prices after the move, and r the short rate.

10 Example 4: finding p, then pricing the call

One-year zero: 95.0908 today; 97.0874 up, 98.0392 down. Rate today 5%

Grow today's price one step	95.0908 × 1.025	**97.4681**
Numerator	98.0392 − 97.4681	**0.5711**
Denominator	98.0392 − 97.0874	**0.9518**
Risk-neutral p	0.5711 ÷ 0.9518	**0.6000**
Call value	(0.6×0 + 0.4×0.5392) ÷ 1.025	**0.2104**

IN THE LESSON

- The one year zero trades at ninety five point zero nine zero eight, and will be worth ninety seven point zero eight seven four or ninety eight point zero three nine two; today's rate is five per cent.
- Grow today's price by one step: times one point zero two five gives ninety seven point four six eight one.
- The numerator is ninety eight point zero three nine two minus that, point five seven one one.
- The denominator is ninety eight point zero three nine two minus ninety seven point zero eight seven four, point nine five one eight.
- So p is point six, the figure the tree used.
- The call is then point four times point five three nine two, over one point zero two five: point two one zero four, the same price replication gave.

11 What the risk-neutral probabilities are

- Not forecasts: they make traded bonds price correctly
- They embed investors' attitude to risk
- Same price for every investor who accepts no arbitrage
- Use the same p for every instrument on the tree
- p outside 0 to 1 means arbitrage. Price 97.60: p ≈ −2.1

IN THE LESSON

- The probabilities are not forecasts.
- They are backed out of market prices so that the traded bonds price correctly, and so they embed investors' attitude to risk.
- So every investor who accepts no arbitrage gets the same price, whatever each believes about rates.
- Use the same p for every instrument on the tree.
- If the implied p lies outside zero to one, the quoted prices admit an arbitrage.
- Suppose the one year zero were quoted at ninety seven point six zero.
- Grown one step it reaches one hundred point zero four, above both future prices, so p comes out at about minus two point one.
- Sell the bond, invest the cash, and win in both states.

12 True probabilities versus risk-neutral ones

- True odds: how rates move in the real world
- Risk-neutral odds: every bond earns the short rate
- Investors demand a premium for bearing rate risk
- Risk-neutral: more weight on rates up, and a higher drift
- Price risk-neutral; forecast and report risk with true

IN THE LESSON

- True probabilities describe how rates move in the real world.
- Risk neutral probabilities price bonds as if every bond earned the short rate on average.
- They differ because investors demand a premium for holding interest rate risk.
- A bond holder is hurt when rates rise and prices fall, so the risk neutral tree puts more weight on rates up than the true odds do, and its drift of rates, their expected change, is higher than the true drift.
- Price with the risk neutral tree.
- Forecast, and report risk, with the true drift.
- Never mix them: a real world expected payoff discounted at the riskless rate misprices.

13 Example 5: two drifts on one tree

Economists: true odds 50:50. Pricing tree: p = 0.6. Rates 6% or 4%

True expected rate, six months	0.5 × 6% + 0.5 × 4%	**5.0%**
Risk-neutral expected rate	0.6 × 6% + 0.4 × 4%	**5.2%**
True drift over the step	5.0% − 5.0%	**0 bp**
Risk-neutral drift	5.2% − 5.0%	**+20 bp**
Drift gap: pays for rate risk	20 − 0	**20 bp**

IN THE LESSON

- Larkfield's economists think an up move and a down move are equally likely, so the true probabilities are fifty fifty, while the pricing tree uses point six for up.
- Under the true odds, the expected rate in six months is half of six plus half of four, five per cent.
- Under the risk neutral odds it is point six times six plus point four times four, five point two per cent.
- Today's rate is five per cent.
- So the true drift over the step is zero, while the risk neutral drift is plus twenty basis points, where a basis point is one hundredth of one per cent.
- The gap of twenty basis points is the extra drift that pays investors for bearing rate risk.

14 Example 6: the market price of risk

Lambda = (expected bond return − short rate) ÷ volatility. One-year zero, six months

Return if rates rise	97.0874 ÷ 95.0908 − 1	**2.0996%**
Return if rates fall	98.0392 ÷ 95.0908 − 1	**3.1006%**
True expected return	half of each	**2.6001%**
Volatility of return	(3.1006 − 2.0996) ÷ 2	**0.5005%**
Market price of risk, lambda	(2.6001 − 2.5000) ÷ 0.5005	**0.20**

At p = 0.6 the expected return is 2.5%, exactly the short rate

IN THE LESSON

- The gap is measured by the market price of risk, lambda: roughly a bond's expected return minus the short rate, divided by the volatility of that return.
- Take the one year zero over six months.
- If rates rise it returns two point zero nine nine six per cent; if they fall, three point one zero zero six.
- At the true fifty fifty odds the expected return is the average, two point six zero zero one per cent.
- The volatility is half the gap between the returns, point five zero zero five per cent.
- The six month short rate is two point five per cent, so lambda is point one zero zero one over point five zero zero five, about point two.
- Check: point two times the rate's one point step volatility is Example five's twenty basis point drift gap.
- At p of point six, the expected return is exactly two point five per cent, the short rate.

15 Many periods: repeat the argument node by node

- At the last date, the payoff is known at each node
- Step back: replicate, or use the node's rate and p
- Repeat back to the root
- Dynamic: rebalance at every node; exact only if costless
- Early exercise: keep the larger of exercise and continuation

IN THE LESSON

- The one period argument extends to many periods, node by node.
- At the last date the derivative's payoff is known at every node.
- Step back one date.
- At each node, either build the replicating portfolio of the underlying bond and cash, or, equivalently, take the risk neutral average and discount it at that node's own rate.
- Repeat back to the root.
- Replication is dynamic: the portfolio is rebalanced at every node, so the price is exact only if rebalancing costs nothing.
- For an option that can be exercised early, compare at each node the value of exercising now with the value of continuing, and keep the larger.

16 Example 7: the 18-month zero on the tree

A zero pays $100 at 1.5 years. Rates at one year: 7%, 5%, 3%. p = 0.6

One year, rate 7%	100 ÷ 1.035	**96.6184**
One year, rates 5% and 3%	100 ÷ 1.025; 100 ÷ 1.015	**97.5610; 98.5222**
Six months, up (6%)	(0.6×96.6184+0.4×97.5610)÷1.03	**94.1703**
Six months, down (4%)	(0.6×97.5610+0.4×98.5222)÷1.02	**96.0250**
Today (5%)	(0.6×94.1703+0.4×96.0250)÷1.025	**92.5972**

IN THE LESSON

- Next, a zero paying one hundred dollars at eighteen months.
- At one year, each node discounts it six months at its own rate.
- At seven per cent that gives ninety six point six one eight four.
- At five and three per cent, ninety seven point five six one zero and ninety eight point five two two two.
- At the six month up node, point six times ninety six point six one eight four plus point four times ninety seven point five six one zero, over one point zero three, gives ninety four point one seven zero three.
- At the down node, the middle and bottom values over one point zero two give ninety six point zero two five zero.
- At the root, over one point zero two five, ninety two point five nine seven two.

17 Example 8: a one-year call on the 18-month zero

European call, strike 97.00, expires at one year. Step back with p = 0.6

Payoffs at one year	bond value − 97, if positive	**0; 0.5610; 1.5222**
Six months, up (6%)	(0.6×0 + 0.4×0.5610) ÷ 1.03	**0.2179**
Six months, down (4%)	(0.6×0.5610+0.4×1.5222)÷1.02	**0.9269**
Today (5%)	(0.6×0.21785+0.4×0.92691)÷1.025	**0.4892**
Wrong: each path at today's rate	0.5128 ÷ 1.025^2	**0.4881**

IN THE LESSON

- Now a European call on that zero, strike ninety seven, exercisable only at one year.
- The payoffs are the bond values less ninety seven: zero at the top node, point five six one zero in the middle, one point five two two two at the bottom.
- At the six month up node, point four times point five six one zero, over one point zero three, is point two one seven nine.
- At the down node, over one point zero two, point nine two six nine.
- Today, over one point zero two five, the call is worth point four eight nine two.
- The wrong method takes the expected payoff over the paths, point five one two eight, and discounts it at today's rate for both steps, giving point four eight eight one.

18 Example 9: the copy must be rebalanced

Delta = change in call value ÷ change in bond value, at each node

Today	−0.7091 ÷ −1.8547	**0.3823 bonds**
After an up move	−0.56098 ÷ −0.94262	**0.5951 bonds**
After a down move	−0.9612 ÷ −0.9612	**1.0000 bond**
Rebalance after up	0.5951 − 0.3823	**+0.2128 bonds**
Rebalance after down	1.0000 − 0.3823	**+0.6177 bonds**

IN THE LESSON

- Why is replication dynamic?
- Delta, the change in the call's value over the change in the bond's value, differs at each node.
- From today, the call will be worth point two one seven nine or point nine two six nine, and the zero ninety four point one seven zero three or ninety six point zero two five zero.
- Delta is minus point seven zero nine one over minus one point eight five four seven: point three eight two three bonds.
- After an up move, minus point five six zero nine eight over minus point nine four two six two gives point five nine five one.
- After a down move, minus point nine six one two over minus point nine six one two is exactly one, because the call will surely be exercised.
- So after an up move the desk buys point two one two eight more bonds, and after a down move point six one seven seven more.

19 The option-adjusted spread

- OAS: constant spread added to every rate on the tree
- Chosen so the model price equals the market price
- Pays for credit, liquidity, model risk once options are valued
- Callable: OAS below Z-spread. Putable: OAS above
- Positive OAS: cheap versus the model. Same model only

IN THE LESSON

- The option adjusted spread, O A S, is the constant spread added to every short rate on the tree so that the model price equals the market price.
- It is the compensation for credit, liquidity and model risk left once any embedded option has been valued with the model.
- Compare the Z spread, a static spread over the zero curve that ignores the option.
- For a callable bond, which the issuer may repay early, the O A S is below the Z spread; for a putable bond, which the holder may sell back, it is above.
- A positive O A S means the bond looks cheap against the model, but compare O A S only across bonds valued with the same model.

20 Example 10: finding an OAS on the tree

A one-year corporate zero trades at 94.81. Tree price at zero spread: 95.0908

Try 30 bp: up node, 6.3%	100 ÷ 1.0315	**96.9462**
Down node, 4.3%	100 ÷ 1.0215	**97.8953**
Root, 5.3%	(0.6×96.9462+0.4×97.8953)÷1.0265	**94.8133**
Model price against market	94.8133 ≈ 94.81	**match to the cent**
Option-adjusted spread	the spread that matches	**about 30 bp**

IN THE LESSON

- A company's one year zero trades at ninety four point eight one.
- At zero spread the tree prices it like the government zero, ninety five point zero nine zero eight, too high, so the spread must be positive.
- Try thirty basis points on every rate.
- At the up node the rate becomes six point three per cent, giving ninety six point nine four six two.
- At the down node, four point three per cent gives ninety seven point eight nine five three.
- At the root, now five point three per cent, we get ninety four point eight one three three.
- That matches the market price to the cent, so the O A S is about thirty basis points.
- In practice the search is done by iteration.

21 Example 11: OAS of a callable bond, the quick way

Callable bond: model price at zero spread 103.40, market 101.00, duration 4.0

Approximate OAS	(103.40 − 101.00) ÷ (101 × 4.0)	**0.594% = 59 bp**
Z-spread, option ignored	from the zero curve	**90 bp**
Option cost	Z-spread − OAS = 90 − 59	**31 bp**
Higher volatility: model price	the call is worth more	**102.60**
OAS at higher volatility	(102.60 − 101.00) ÷ (101 × 4.0)	**0.396% = 40 bp**

IN THE LESSON

- For a callable bond, a rule of thumb gives the O A S quickly; it is only an approximation of the tree search.
- O A S is about P zero minus P, over P times D: the model price at zero spread, the market price, and the effective duration, the model's percentage price change for a one point move in rates.
- Here that is two point four over one hundred and one times four, about point five nine four per cent, fifty nine basis points.
- The Z spread, which ignores the call, is ninety basis points, so the option cost is thirty one.
- Now assume higher volatility.
- The call is worth more to the issuer, so the model price falls to one hundred and two point six zero, and the O A S falls to about forty basis points.

22 Why trees should recombine

- Not recombining: step n has 2^n nodes
- Recombining: up-down = down-up, so step n has n + 1
- Total through n steps: (n + 1)(n + 2) ÷ 2
- Polynomial growth makes small time steps feasible
- Constant additive or multiplicative moves: 5+1−1 = 5−1+1

IN THE LESSON

- In Larkfield's tree, the middle node at one year is reached by up then down, and by down then up.
- In a non recombining tree those two orders lead to different nodes, so step n has two to the power n nodes, and the tree grows exponentially.
- In a recombining tree they meet: up then down equals down then up, so step n has only n plus one nodes.
- The total through n steps, counting the start, is n plus one times n plus two, divided by two.
- This polynomial growth makes small time steps feasible.
- That needs constant additive or multiplicative moves.
- Ours are additive: five plus one minus one equals five minus one plus one.

23 Counting the nodes

Nodes at the last date, and in total for a recombining tree

Steps n	Not recombining	Recombining	Total
2	4	3	6
10	1,024	11	66
20	1,048,576	21	231

IN THE LESSON

- With two steps, as in Larkfield's tree, a non recombining tree has four nodes at the last date against three, and the recombining tree has six in all: three times four over two.
- With ten steps, one thousand and twenty four against eleven, with sixty six in total.
- With twenty steps, ten years in six month steps, the non recombining tree has one million, forty eight thousand, five hundred and seventy six nodes at the last date alone; the recombining tree has twenty one, and two hundred and thirty one in all.
- The limit is path dependence: a mortgage cash flow that depends on past prepayments, and so on the route rates took, cannot be valued on a plain recombining tree.

24 Constant-maturity Treasury swaps

- Swap a fixed rate K for a Treasury par yield, e.g. 2-year
- Pay = N × (CMT − K) ÷ 2 for semiannual payments
- First find the CMT rate at each node
- Discount backward with the short rate and p
- Set now, paid in six months: one more step at that node

IN THE LESSON

- A constant maturity Treasury swap, or C M T swap, swaps a fixed rate, K, for a floating rate set at a Treasury par yield of fixed maturity, for example the two year yield, rather than at the short rate.
- For semiannual payments on notional N, the payment at a node is N times C M T minus K, divided by two.
- To price it on a tree with risk neutral probabilities, first work out the C M T rate at each node.
- Then discount backward node by node with the short rate and the risk neutral p.
- A payment set at one date and paid six months later is discounted one more step, at the rate of the node where it is set.

25 Example 12: the swap's payments

Receive CMT, pay 5% on $10m. CMT: 6.3%, 4.3%; then 7.2%, 5.2%, 3.2%

Six months, up: CMT 6.3%	10m × (6.3% − 5%) ÷ 2	**+$65,000**
Six months, down: CMT 4.3%	10m × (4.3% − 5%) ÷ 2	**−$35,000**
One year, top: CMT 7.2%	10m × 2.2% ÷ 2	**+$110,000**
One year, middle: CMT 5.2%	10m × 0.2% ÷ 2	**+$10,000**
One year, bottom: CMT 3.2%	10m × (−1.8%) ÷ 2	**−$90,000**

IN THE LESSON

- Larkfield receives the two year C M T rate and pays five per cent on ten million dollars.
- Rates are set at six months and one year, each paid six months after it is set.
- The two year par yields are six point three and four point three per cent at six months, then seven point two, five point two and three point two per cent.
- At the six month up node, ten million times one point three per cent over two is a receipt of sixty five thousand dollars.
- At the down node, minus point seven per cent means paying thirty five thousand.
- At one year, the top node receives one hundred and ten thousand, the middle ten thousand, and the bottom pays ninety thousand.

26 Example 13: one step back from one year

Each payment is paid six months after it is set: one more step at that node

One year, top (7%)	110,000 ÷ 1.035	**106,280**
One year, middle (5%)	10,000 ÷ 1.025	**9,756**
One year, bottom (3%)	−90,000 ÷ 1.015	**−88,670**

IN THE LESSON

- First discount each payment one step at the node where it is set.
- At the top node, one hundred and ten thousand over one point zero three five is one hundred and six thousand two hundred and eighty.
- The middle gives ten thousand over one point zero two five, nine thousand seven hundred and fifty six, and the bottom minus ninety thousand over one point zero one five, minus eighty eight thousand six hundred and seventy.

27 Example 13: the six-month nodes

Six-month node: average of next values, plus its own payment, at its rate

Up (6%): average of next values	(0.6×106,280+0.4×9,756)÷1.03	**65,699.4**
Up: add its own payment	65,000 ÷ 1.03 + 65,699.4	**128,806**
Down (4%): average of next values	(0.6×9,756−0.4×88,670)÷1.02	**−29,033.7**
Down: add its own payment	−35,000 ÷ 1.02 − 29,033.7	**−63,347**

IN THE LESSON

- Each six month node adds its own payment to the risk neutral average of the next values.
- At the up node, point six times one hundred and six thousand two hundred and eighty plus point four times nine thousand seven hundred and fifty six, over one point zero three, is sixty five thousand six hundred and ninety nine point four.
- Add its own payment, sixty five thousand over one point zero three: one hundred and twenty eight thousand eight hundred and six.
- At the down node, point six times nine thousand seven hundred and fifty six minus point four times eighty eight thousand six hundred and seventy, over one point zero two, is minus twenty nine thousand and thirty three point seven.
- Add its own payment, minus thirty five thousand over one point zero two: minus sixty three thousand three hundred and forty seven.

28 Example 14: the value today, and the trap

Up node 128,806; down node −63,347. Root rate 5%; p = 0.6

Swap value today (net)	(0.6×128,806−0.4×63,347)÷1.025	**$50,678**
Fixed leg: each payment	10m × 5% ÷ 2	**$250,000**
Fixed leg value	250,000 × (0.950908 + 0.925972)	**$469,220**
CMT leg alone	50,678 + 469,220	**$519,898**
Wrong: forward CMT, today's DFs	rate and discounting correlated	**convexity error**

IN THE LESSON

- Point six times the up value, minus point four times sixty three thousand three hundred and forty seven, over one point zero two five, gives fifty thousand six hundred and seventy eight dollars, the net value of the swap to Larkfield, since each payment nets the fixed rate.
- As a check, value the fixed leg alone.
- Each fixed payment is ten million times five per cent over two, two hundred and fifty thousand dollars, paid at one year and eighteen months, valued with the zero prices from Examples one and seven, point nine five zero nine zero eight and point nine two five nine seven two: four hundred and sixty nine thousand two hundred and twenty dollars.
- So the C M T leg alone is worth fifty thousand six hundred and seventy eight plus that, five hundred and nineteen thousand eight hundred and ninety eight.
- The trap is to discount forward C M T rates with today's discount factors; the rate and the discount factor move together, so that misses a convexity effect.

29 Smaller time steps: what is gained

- Better approximation of continuous rate movements
- More possible rates at each date
- Finer exercise dates for options
- Prices converge toward the continuous-time value
- But a smaller step does not repair a wrong model

IN THE LESSON

- What does the desk gain by shrinking the step from six months to one month?
- The tree becomes a better approximation of rates that really move continuously.
- There are more possible rates at each date, so the spread of future rates is captured more finely.
- Options get finer exercise dates, closer to the real contract.
- And prices converge toward the value of the continuous time model.
- A smaller step improves the numerics, the arithmetic of the approximation, but it does not repair a wrong model.
- If the model's volatility or drift is wrong, a finer tree only gives the wrong answer more precisely.

30 Example 15: what a finer tree costs

Annual volatility = 1% ÷ root 0.5 = 1.414%. Per step: annual × root(step in years)

Six-month step size	1.414% × root 0.5	**1.00%**
One-month step size	1.414% × root(1/12)	**0.41%**
One-month discounting at 5%	1 + 0.05 ÷ 12	**1.004167**
10 years in 20 steps: nodes	21 × 22 ÷ 2	**231**
10 years in 120 steps: nodes	121 × 122 ÷ 2	**7,381**

About n^2 ÷ 2 nodes: 6 times the steps, about 32 times the work

IN THE LESSON

- The costs are computation and calibration.
- Per step volatility is the annual volatility times the square root of the time step, in years.
- Our tree moves the rate one point every six months, so annual volatility is one per cent over root point five, about one point four one four per cent, and parameters must be rescaled with the step.
- A six month step uses one point four one four times root point five: one per cent.
- A one month step uses root one twelfth: about point four one per cent.
- Monthly discounting at five per cent uses one plus point zero five over twelve, one point zero zero four one six seven.
- Ten years in twenty steps needs two hundred and thirty one nodes; in one hundred and twenty monthly steps, seven thousand three hundred and eighty one.
- Nodes grow like n squared over two: six times the steps, about thirty two times the work.

31 Black-Scholes-Merton and bonds

- BSM: constant volatility, lognormal price, constant rate
- Bond volatility shrinks to zero near maturity: pull to par
- Lognormal price lets a zero rise above par: negative rates
- Driving rate also discounts: cannot be held constant
- Tolerable: short option on a long bond. Else a rate model

IN THE LESSON

- Can the desk skip the tree and use Black Scholes Merton?
- It assumes the underlying has a constant volatility and a lognormal price, and that the riskless rate is constant.
- For fixed income all three fail.
- A bond's price volatility shrinks to zero as it approaches par at maturity, the pull to par, so constant volatility is wrong.
- A lognormal price lets a zero trade above par, which implies a negative rate.
- And the short rate that drives the bond is also the rate used for discounting, so holding it constant is inconsistent.
- The model is tolerable only when the option is short dated relative to a long maturity bond.
- For options on rates, or long dated options on short bonds, use a term structure model.

32 Example 16: where BSM breaks

Price volatility ≈ duration × yield volatility. Yield volatility 1%

Zero at 98, 5% vol: up 2 sd	98 × e^{0.10}	**108.31 > par**
3-month option, 10-year zero	duration 10 to 9.75	**10% to 9.75%**
4-year option, 5-year zero	duration 5 to 1	**5% to 1%**
Verdict, short option	volatility nearly constant	**BSM tolerable**
Verdict, long option	volatility collapses	**use a rate tree**

IN THE LESSON

- Before using the model, convert yield volatility to price volatility with duration: price volatility is roughly duration times yield volatility, here one per cent.
- First, a zero at ninety eight with five per cent price volatility.
- A lognormal move two standard deviations up multiplies it by e to the power point one: one hundred and eight point three one, above par, a negative rate.
- Second, a three month option on a ten year zero.
- Its duration falls only from ten to nine point seven five, so price volatility barely moves, from ten to nine point seven five per cent: the model is tolerable.
- Third, a four year option on a five year zero.
- Duration falls from five to one, so volatility collapses from five per cent to one.
- Use a rate tree.

33 Try three quick questions

1. Node rate 6%. Next values 98 (up) and 99 (down); p = 0.5. Node value?
   Answer: (0.5×98 + 0.5×99) ÷ 1.03 = 95.63.
2. Bond 95.50 today; 97 if rates rise, 98 if they fall; rate 4%. Risk-neutral p?
   Answer: (98 − 95.50 × 1.02) ÷ (98 − 97) = 0.59.
3. Callable: model price at zero spread 100.80, market 100.00, duration 5. OAS?
   Answer: About 0.80 ÷ (100 × 5) = 0.16% = 16 bp.

IN THE LESSON

- Pause after each question, then listen to the answer.
- Question one.
- A node's rate is six per cent, the next values are ninety eight if rates rise and ninety nine if they fall, and p is point five.
- Its value?
- The average, ninety eight point five, divided by one point zero three: ninety five point six three.
- Question two.
- A bond is worth ninety five point five zero today, ninety seven if rates rise and ninety eight if they fall, and the rate is four per cent.
- The risk neutral p?
- Ninety eight minus ninety five point five times one point zero two, over ninety eight minus ninety seven: point five nine.
- Question three.
- A callable bond has a model price at zero spread of one hundred point eight, a market price of one hundred and a duration of five.
- Its rough O A S?
- Point eight over five hundred: sixteen basis points.