CHARTA LEARNING · LESSON NOTES

VaR Mapping

FRM Part II

1 A story: twelve million numbers

- Halden's desk: about 5,000 positions
- Full matrix: 5,000 × 5,001 ÷ 2 ≈ 12.5m numbers
- Short or no price histories: impractical and noisy
- 30 factors: 30 × 31 ÷ 2 = 465 numbers
- Mapping: describe each position by factor exposures

IN THE LESSON

- Halden Bank's markets desk holds about five thousand positions.
- To compute V a R directly, the risk team would need a variance for every position and a covariance for every pair.
- That is five thousand times five thousand and one, divided by two, about twelve and a half million numbers.
- Many positions have short price histories, or none, so this is impractical and noisy.
- With thirty market factors instead, the matrix needs only thirty times thirty one, divided by two, four hundred and sixty five numbers.
- That is mapping: describe each position by factor exposures.

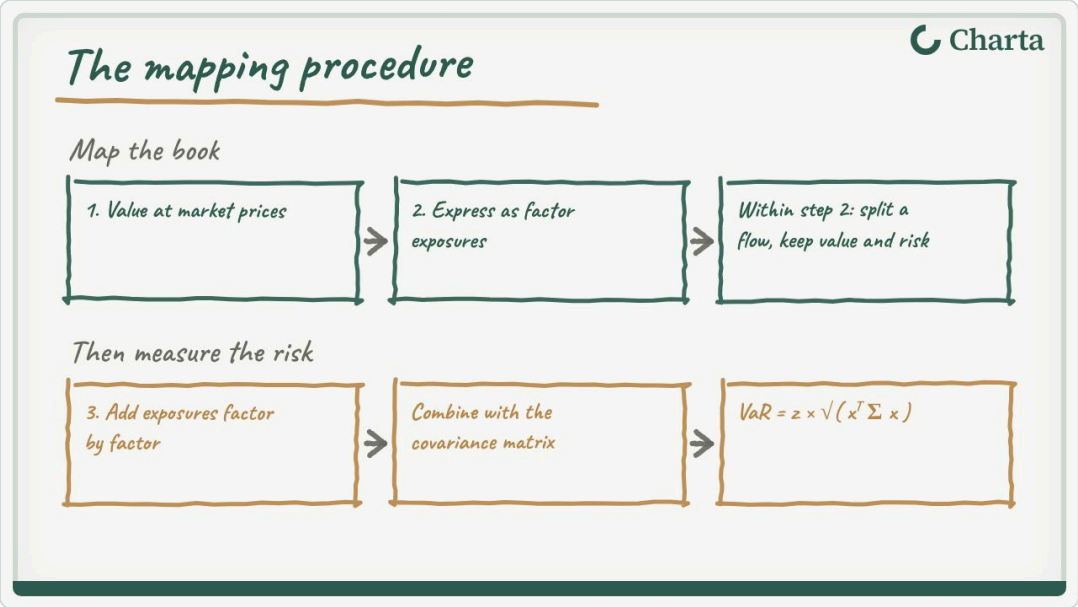
2 Primitive risk factors

- Primitive risk factors: a few basic, shared market prices
- Equity indices and exchange rates
- Zero-coupon yields at standard maturities: vertices
- Share: index. Bond: vertices. Forward: FX, two rates
- We lose some accuracy and gain tractability

IN THE LESSON

- Mapping replaces each position with exposures to a few primitive risk factors, basic prices that many positions share.
- Typical choices are equity indices and exchange rates, and zero coupon yields at standard maturities, which are called vertices.
- A share maps to its index, a bond to nearby vertices, and a currency forward to an exchange rate and two interest rates.
- We lose some accuracy, since no short list of factors captures every detail, but gain tractability: a covariance matrix small enough to estimate well.

3 The mapping procedure



IN THE LESSON

- The procedure has three steps, in order.
- The first row maps the book.
- Step one: value every position at current market prices.
- Step two: express each position as exposures to the chosen risk factors.
- Within step two, a cash flow that falls between two vertices is split between them, so that both its market value and its market risk are kept.
- The second row then measures the risk.
- Step three: add the exposures factor by factor into one exposure vector.
- Combine that vector with the factor covariance matrix.
- The result is V a R, z times the square root of x transpose sigma x.
- The exam point: map first, then measure risk.

4 Portfolio VaR from the mapped exposures

VaR = z × √(x^{T}Σx)
z = 1.65 (95%, one-tailed); horizon one month

z, the normal quantile	Multiplier for the confidence
Exposure vector, x	Money mapped to each factor
Factor covariance matrix, Σ	Factor variances and covariances

IN THE LESSON

- Write z for the normal quantile, the multiplier for the confidence level.
- Throughout this lesson z is one point six five, the ninety five per cent one tailed value, rounded, and the horizon is one month.
- Write x for the exposure vector, the money mapped to each factor, and capital sigma for the factor covariance matrix, holding each factor's variance and each pair's covariance.
- Then x transpose sigma x is the variance of the portfolio's value, and V a R is z times its square root.
- Sigma's size depends on the number of factors, not positions.

5 Example 1: splitting a cash flow

A $5m cash flow due in 2.5 years. Vertices at 2 and 3 years; monthly σ

Vertex volatilities	2 yr 0.55%, 3 yr 0.85%	**ρ = 0.97**
Volatility at 2.5 years	(0.55 + 0.85) ÷ 2	**0.70%**
Keep the value	α at 2 yr, 1 − α at 3 yr	**total $5m**
Keep the risk	variance of the mix = 0.70^2	**solve for α**

IN THE LESSON

- Halden holds a note with one cash flow, due in two and a half years, worth five million dollars today, between vertices at two and three years.
- Their monthly price volatilities are point five five per cent at two years and point eight five per cent at three, with a correlation of point nine seven.
- First, interpolate the volatility at two and a half years: point five five plus point eight five, over two, is point seven per cent.
- Next, keep the value: put a share alpha of the five million at two years and one minus alpha at three years.
- Last, keep the risk: choose alpha so that the variance of the mix equals point seven squared.

6 Example 1, continued: solving for alpha

Solve $\alpha^2(0.55)^2 + (1-\alpha)^2(0.85)^2 + 2\alpha(1-\alpha)(0.97)(0.55)(0.85) = 0.49$

Coefficient of α^2	1.025 − 0.90695	**0.11805**
Coefficient of α	−2 × 0.7225 + 0.90695	**−0.53805**
Constant	0.7225 − 0.49	**0.2325**
Root between 0 and 1	the other root is 4.07	**α = 0.4834**
Mapped amounts	5 × 0.4834; 5 × 0.5166	**$2.42m; $2.58m**

IN THE LESSON

- The variance of the mix is alpha squared times point five five squared, plus one minus alpha, squared, times point eight five squared, plus twice alpha times one minus alpha times the correlation and both volatilities.
- Set it equal to point four nine, in per cent squared, and collect the terms.
- The alpha squared coefficient is one point zero two five, minus point nine zero six nine five: point one one eight zero five.
- The coefficient of alpha is minus two times point seven two two five, plus point nine zero six nine five: minus point five three eight zero five.
- The constant is point seven two two five minus point four nine, point two three two five.
- The only root between zero and one is point four eight three four; the other is four point zero seven.
- So two point four two million goes to two years, and two point five eight million to three.

7 Two layers: general and specific risk

- General risk: the part driven by the common factors
- Stock mapped to an index: systematic $\sigma = \beta \times \sigma_M$
- Specific risk: the residual, idiosyncratic part
- Betas add; residuals are uncorrelated and diversify
- Hedging beta removes general risk, not specific risk

IN THE LESSON

- Mapping also shows two layers of risk.
- General, or market wide, risk is the part of a position's variance driven by the common factors it is mapped to.
- For a stock mapped to an index, its systematic volatility is beta times the index volatility, where beta measures how strongly the stock moves with the index.
- Specific risk is the residual, idiosyncratic part, such as news about the company alone.
- Beta exposures add up in a straight line, but residuals are assumed uncorrelated, so specific risk diversifies away in a large portfolio.
- And hedging the beta removes general risk, but not specific risk.

8 General VaR and specific VaR

$\sigma^2 = \beta^2 \times \sigma_M^2 + \sigma_R^2$
$GVaR = z \times \beta \times \sigma_M \times V$ $SVaR = z \times \sigma_R \times V$
$VaR = \sqrt{(\ GVaR^2 + SVaR^2\ )}$

Beta to the index, β	How much it moves with the index
Index volatility, σ_M	Common factor volatility
Residual volatility, σ_R	Idiosyncratic volatility
Position value, V	Money in the stock

IN THE LESSON

- Write sigma M for the index volatility, sigma R for the residual volatility, beta for the stock's beta, and V for the position value.
- One stock's total variance is beta squared times sigma M squared, plus sigma R squared.
- The general V a R is z times beta times sigma M times V.
- The specific V a R uses only the residual volatility, with no beta at all: it is z times sigma R times V.
- Because the two parts are assumed to be uncorrelated, never add them.
- The total V a R is the square root of general V a R squared plus specific V a R squared.

9 Example 2: one stock

Corvane Motors: $6m, β = 1.2, index σ_M = 5%, residual σ_R = 8%, monthly

Total variance	1.44 × 0.0025 + 0.0064	**0.0100**
Total volatility	√0.0100	**10%**
General VaR	1.65 × 1.2 × 5% × $6m	**$594,000**
Specific VaR	1.65 × 8% × $6m	**$792,000**
Total VaR	√(594² + 792²), in $000	**$990,000**

Check: 1.65 × 10% × $6m = $990,000. Never add: 594 + 792 = 1,386

IN THE LESSON

- Halden holds six million dollars of Corvane Motors: beta one point two, index volatility five per cent a month, residual volatility eight per cent.
- The total variance is one point four four times point zero zero two five, plus point zero zero six four, which is point zero one, so the total volatility is ten per cent.
- General V a R is one point six five times one point two times five per cent times six million, five hundred and ninety four thousand dollars.
- Specific V a R is one point six five times eight per cent times six million, seven hundred and ninety two thousand.
- The total is the square root of their squares added, nine hundred and ninety thousand dollars.
- The check, one point six five times ten per cent times six million, agrees.
- Adding them gives one point three eight six million dollars, which is too much.

10 Example 3: two stocks, and a beta hedge

Add Tessaly Foods: $4m, β = 0.9, σ_R = 6%. Residuals are uncorrelated

Beta exposures add	6 × 1.2 + 4 × 0.9	**$10.8m**
General VaR	1.65 × 5% × $10.8m	**$891,000**
Specific VaR	1.65 × √(0.48² + 0.24²)	**$885,500**
Total VaR	√(891² + 885.5²), in $000	**$1,256,200**
Short $10.8m of index futures	general VaR → 0; total = SVaR	**$885,500**

IN THE LESSON

- Now add four million dollars of Tessaly Foods, with a beta of point nine and a residual volatility of six per cent.
- Beta exposures add: six times one point two plus four times point nine, ten point eight million dollars.
- General V a R is one point six five times five per cent times ten point eight million, eight hundred and ninety one thousand.
- The residual risks, point four eight and point two four million, are uncorrelated, so specific V a R uses the root of their summed squares: about eight hundred and eighty five thousand five hundred.
- Total V a R is about one million two hundred and fifty six thousand two hundred dollars.
- If Halden sells ten point eight million of index futures, general V a R goes to zero, so total V a R falls to the specific V a R, about eight hundred and eighty five thousand five hundred.

11 Three ways to map a bond

Accuracy: cash flow first, then duration, then principal

Method	Where the value goes	Verdict
Principal	All at final maturity	Overstates risk
Duration	All at the duration	Parallel shifts only
Cash flow	Each PV to its vertices	Most accurate, most data

IN THE LESSON

- There are three standard ways to map a bond onto zero coupon factors, and by accuracy, cash flow mapping comes first, then duration, then principal.
- Principal mapping puts the bond's whole market value at the vertex of its final maturity.
- It is simplest, but overstates the risk of a coupon bond, because coupons shorten its effective maturity.
- Duration mapping puts the whole value at the vertex equal to the bond's duration, interpolating if needed.
- It captures the coupon effect in one number, but sees only parallel shifts.
- Cash flow mapping discounts every payment at its own spot rate and allocates each present value to neighbouring vertices.
- It is the most accurate, and needs the most data.

12 Example 4: Halden's three-year bond

$10m, 3-year bond, 5% annual coupon. Spot rates 4%, 4.5%, 5%. In $000

Year 1 coupon	500 ÷ 1.04	**481**
Year 2 coupon	500 ÷ 1.045^2	**458**
Year 3 coupon and principal	10,500 ÷ 1.05^3	**9,070**
Market value	481 + 458 + 9,070	**10,009**
Duration, years	(481+2×458+3×9,070)÷10,009	**2.858**

IN THE LESSON

- Halden holds a three year, ten million dollar bond with a five per cent annual coupon.
- Spot rates are four, four and a half and five per cent for one, two and three years.
- In thousands of dollars, the first coupon of five hundred, discounted at four per cent, is worth four hundred and eighty one.
- The second, discounted twice at four and a half per cent, is worth four hundred and fifty eight.
- The final ten thousand five hundred, discounted three times at five per cent, is worth nine thousand and seventy.
- The market value is ten thousand and nine.
- Duration is four eighty one, plus two times four fifty eight, plus three times nine thousand and seventy, over ten thousand and nine, which is two point eight five eight years.

13 Example 5: the three methods compared

Monthly σ: 1y 0.20%, 2y 0.55%, 3y 0.85%. ρ(1,2) 0.95, ρ(1,3) 0.90, ρ(2,3) 0.97. In $000

Principal: all at 3 years	1.65 × 0.85% × 10,009	**$140,400**
σ at 2.858 years	0.55 + 0.858 × 0.30	**0.807%**
Duration VaR	1.65 × 0.807% × 10,009	**$133,300**
Cash flow: d = PV × σ	0.96, 2.52, 77.10	**variance 6,466**
Cash-flow VaR	1.65 × √6,466 = 1.65 × 80.41	**$132,700**

Principal overstates the most; cash flow mapping is the most accurate

IN THE LESSON

- Monthly vertex volatilities are point two, point five five and point eight five per cent, with correlations of point nine five, point nine and point nine seven, and figures are in thousands.
- Principal mapping puts all ten thousand and nine at three years: V a R is about one hundred and forty thousand four hundred dollars.
- Duration mapping interpolates the volatility at two point eight five eight years: point five five plus point eight five eight times point three, point eight zero seven per cent, so V a R is about one hundred and thirty three thousand three hundred.
- Cash flow mapping keeps three money volatilities, present value times volatility: nine hundred and sixty, two thousand five hundred and twenty, and seventy seven thousand one hundred dollars.
- With the correlations, the variance is about six thousand four hundred and sixty six.
- Its root is eighty point four one, so V a R is about one hundred and thirty two thousand seven hundred.
- Principal mapping overstates the most; cash flow mapping is the most accurate.

14 A bond book through benchmark instruments

- Benchmarks: zero-coupon bonds at standard vertices
- For example 3 months; 1, 2, 3, 5, 7, 10 and 30 years
- Each cash flow: PV, then allocate to nearest vertices
- Sum vertex by vertex; short positions enter as negatives
- One net vector: where risk sits and what hedges it

IN THE LESSON

- Now scale up to Halden's whole bond book, expressed through a set of benchmark instruments, typically zero coupon bonds at standard vertices: for example three months, and one, two, three, five, seven, ten and thirty years.
- Each position's cash flows are discounted to present value and allocated to the nearest vertices, keeping value and risk unchanged.
- Then the present values are summed vertex by vertex across all the bonds, with short positions entered as negatives.
- The result is one net exposure vector.
- It shows where the risk is concentrated, and which benchmark instruments could hedge it.

15 VaR from the vertex exposures

VaR = z × √(Σ_i Σ_j x_i × x_j × σ_i × σ_j × ρ_{ij})

Net the signed exposures first, then apply ρ

Net exposure at vertex i	Long positive, short negative
Volatility of vertex i, σ_i	Price volatility of vertex zero
Correlation of i and j, ρ_ij	Below 1, so vertices diversify

IN THE LESSON

- Write x i for the net exposure at vertex i, sigma i for its volatility, and rho i j for the correlation between vertices i and j.
- Multiply x i times x j times sigma i times sigma j times rho i j, sum over every pair of vertices, take the square root, and multiply by z.
- Offsetting positions at the same vertex cancel before any risk is computed, and imperfectly correlated vertices diversify each other.
- So the exam rule is: net the signed exposures first, then apply the correlations.

16 Example 6: Halden's net exposure vector

Bond (Example 4), 2.5-year note (Example 1), short 2-year zero of $8m. In $000

1-year vertex	the bond's first coupon	**+481**
2-year vertex	458 + 2,417 − 8,000	**−5,125**
3-year vertex	9,070 + 2,583	**+11,653**

Short positions enter as negatives: long 3 years, short 2 years

IN THE LESSON

- Halden's bond book holds the three year bond from example four, the two and a half year note from example one, and a short two year zero coupon bond worth eight million dollars.
- In thousands, the one year vertex holds only the bond's first coupon, four hundred and eighty one.
- At two years, the bond's four hundred and fifty eight, plus the note's two thousand four hundred and seventeen, minus the short zero's eight thousand, nets to minus five thousand one hundred and twenty five.
- At three years, the bond's nine thousand and seventy plus the note's two thousand five hundred and eighty three make eleven thousand six hundred and fifty three.
- Short positions enter as negatives, so this book is long three years and short two.

17 Example 7: VaR of the vector

Net vector in $000: 481, −5,125, 11,653. Vertex σ and ρ as in Example 5

1-year: x × σ	481 × 0.20%	**0.96**
2-year: x × σ	−5,125 × 0.55%	**−28.19**
3-year: x × σ	11,653 × 0.85%	**99.05**
Variance	Σ d_i × d_j × ρ_ij	**5,310**
VaR	1.65 × √5,310 = 1.65 × 72.87	**$120,200**

Most risk sits at 3 years: selling 3-year zeros hedges it

IN THE LESSON

- In thousands of dollars, first find each vertex's money volatility, exposure times volatility.
- Four hundred and eighty one times point two per cent is point nine six.
- Minus five thousand one hundred and twenty five times point five five per cent is minus twenty eight point one nine.
- Eleven thousand six hundred and fifty three times point eight five per cent is ninety nine point zero five.
- The variance adds the three squares and twice each pair's product times its correlation: about five thousand three hundred and ten.
- Its square root is seventy two point eight seven, so V a R is about one hundred and twenty thousand two hundred dollars.
- Most of the risk sits at three years, so selling three year zeros hedges it.

18 Stress tests on a mapped book

- Mapping gives stress tests a common language
- Change = Σ exposure × factor shock
- Curve shift or twist, index fall, devaluation, history replay
- Or shock the covariance matrix: volatilities, correlations
- The adverse direction depends on position signs

IN THE LESSON

- Mapping also gives stress testing a common language.
- Once every position is expressed as exposures to a few risk factors, a shock to a factor turns straight into a profit or loss.
- The change in portfolio value is the sum, across factors, of each exposure in money times that factor's shock as a return.
- A manager can shift or twist the curve, drop an index, devalue a currency, or replay history, and revalue the whole book consistently.
- Shocks can also target the covariance matrix itself, for example by raising volatilities or moving correlations.
- Which direction is adverse depends on the signs of the positions.

19 Example 8: a stress scenario

Example 3 stocks (β exposure $10.8m) and the Example 6 bond book. In $000

Equity: index −10%	10,800 × −10%	**−1,080**
1-year zero: price −1%	481 × −1%	**−5**
2-year zero: price −2%	−5,125 × −2%	**+103**
3-year zero: price −3%	11,653 × −3%	**−350**
Total change	sum of the four	**−1,332**

Only the factors move: specific risk is not in this number

IN THE LESSON

- Here is a stress scenario for the stocks of example three, a beta exposure of ten point eight million dollars, and the bond book of example six.
- The equity index falls ten per cent, and zero coupon prices fall one, two and three per cent at one, two and three years.
- In thousands, multiply exposure by shock.
- The equity exposure of ten thousand eight hundred loses one thousand and eighty.
- The one year exposure of four hundred and eighty one loses about five.
- The short two year exposure of minus five thousand one hundred and twenty five gains about one hundred and three.
- The three year exposure of eleven thousand six hundred and fifty three loses about three hundred and fifty.
- The total change is a loss of about one million three hundred and thirty two thousand dollars.
- Only factors move, so specific risk is not in this number.

20 Example 9: a correlation shock

Long 3 years, short 2 years. Shock: ρ(2,3) falls from 0.97 to 0.80

Pair term at ρ = 0.97	2 × 0.97 × (−28.19) × 99.05	**−5,417**
Pair term at ρ = 0.80	2 × 0.80 × (−28.19) × 99.05	**−4,468**
Change in variance	−4,468 − (−5,417)	**+949**
VaR before	Example 7	**$120,200**
VaR after	1.65 × √(5,310 + 949)	**$130,500**

IN THE LESSON

- Halden's bond book is long three years and short two, and the shock cuts their correlation from point nine seven to point eight.
- At point nine seven, the pair term in the variance is two times point nine seven times minus twenty eight point one nine times ninety nine point zero five, about minus five thousand four hundred and seventeen.
- At point eight, it is about minus four thousand four hundred and sixty eight.
- So the variance rises by about nine hundred and forty nine.
- V a R before was one hundred and twenty thousand two hundred dollars.
- After, it is one point six five times the square root of five thousand three hundred and ten plus nine hundred and forty nine, about one hundred and thirty thousand five hundred.
- For a long short pair, a fall in correlation hurts.

21 Limits and strengths of mapped stress tests

- A stress loss comes with no probability attached
- Linear mapping misstates options: it ignores gamma
- It ignores basis risk between position and factor
- Large shocks: delta-gamma or full revaluation
- Not bound by the normal distribution VaR assumes

IN THE LESSON

- Mapped stress tests have limits.
- First, a stress test gives a loss but no probability, so it complements V a R rather than replacing it.
- Second, a linear mapping misstates the result for options, because it ignores gamma, the curvature of an option's value.
- Third, it ignores basis risk, the gap between a position and the factor standing in for it, such as a corporate bond mapped to government zero yields.
- So large shocks call for delta gamma or full revaluation.
- Finally, a stress test reuses the mapping but is not bound by the normal distribution that V a R assumes, so any shock, however extreme, can be tested.

22 Relative VaR against a benchmark

$x = w_P - w_B$ $TEV = \sqrt{x^{T}\Sigma x}$
Relative VaR $= z \times TEV \times V \times \sqrt{T}$
Equal values: $\sqrt{VaR_P^2 + VaR_B^2 - 2\rho \times VaR_P \times VaR_B}$

Active weights, x	Portfolio minus benchmark weights
Factor covariance matrix, Σ	Factor variances and covariances
Portfolio value, V	Money in the portfolio
Horizon, T	In units of the TEV period

IN THE LESSON

- Halden's asset management arm runs the Halden Balanced Fund against a benchmark, and relative V a R measures the risk of falling behind it.
- With portfolio weights w P and benchmark weights w B, the active exposure vector x is w P minus w B.
- The tracking error volatility, T E V, is the square root of x transpose sigma x.
- Relative V a R is z times T E V times the portfolio value V, times the square root of T, the horizon in units of the T E V period.
- When portfolio and benchmark have equal value, with return correlation rho, it is also the square root of V a R P squared plus V a R B squared, minus two rho V a R P V a R B.

23 Example 10: tracking error and relative VaR

Fund $100m: 70/30 equities/bonds vs a 60/40 benchmark. σ 5% and 1.5%, ρ = 0.2

Active weights	70 − 60; 30 − 40	**+10%; −10%**
Variance, in %²	0.25 + 0.0225 − 0.03	**0.2425**
Tracking error volatility	√0.2425	**0.4924%**
Relative VaR, 1 month	1.65 × 0.4924% × $100m	**$812,500**
Relative VaR, 3 months	812,500 × √3	**$1,407,300**

IN THE LESSON

- The one hundred million dollar Halden Balanced Fund holds seventy per cent equities and thirty per cent bonds, against a sixty forty benchmark.
- Monthly volatilities are five per cent for equities and one and a half per cent for bonds, with a correlation of point two.
- The active weights are plus ten per cent and minus ten.
- The variance, in per cent squared, is point two five plus point zero two two five minus point zero three, the cross term: point two four two five, so T E V is point four nine two four per cent a month.
- One month relative V a R, one point six five times that times one hundred million, is about eight hundred and twelve thousand five hundred dollars.
- Over three months, times root three, it is about one million four hundred and seven thousand three hundred.

24 Example 11: absolute and relative risk

Monthly σ^2: benchmark 9 + 0.36 + 0.72 = 10.08; fund 12.25 + 0.2025 + 0.63 ≈ 13.08

Benchmark copy: absolute	1.65 × 3.175% × $100m	**$5.24m**
Benchmark copy: relative	active weights all zero	**$0**
Halden fund: absolute	1.65 × 3.617% × $100m	**$5.97m**
Halden fund: relative	Example 10	**$0.81m**
All cash: absolute; relative	x = −w_B, so TEV = σ_B	**≈ $0; $5.24m**

IN THE LESSON

- From the same matrix, the benchmark's variance is nine plus point three six plus point seven two, ten point zero eight in per cent squared, so its volatility is three point one seven five per cent.
- The fund's is twelve point two five plus point two zero two five plus point six three, about thirteen point zero eight, so its volatility is three point six one seven per cent.
- A fund that copies the benchmark has an absolute V a R of about five point two four million dollars, but a relative V a R of zero: its active weights are all zero.
- The Halden fund's absolute V a R is about five point nine seven million, its relative V a R only point eight one million.
- An all cash fund has almost no absolute V a R, but its active weights are minus the benchmark's, so its relative V a R is the benchmark's five point two four million.

25 Example 12: using the two VaRs

Equal values. VaR_P = $6m, VaR_B = $5m, correlation ρ = 0.9

Squares	$6^2 + 5^2$	**61**
Correlation term	2 × 0.9 × 6 × 5	**54**
Relative VaR	√(61 − 54) = √7	**$2.65m**
Wrong: subtract the VaRs	6 − 5	**$1.00m**
Wrong: ignore correlation	√61	**$7.81m**

Relative VaR is the natural basis for active-risk limits

IN THE LESSON

- With equal values, relative V a R can also come from the two V a Rs and their correlation.
- Take a separate illustration: a portfolio V a R of six million dollars, a benchmark V a R of five million, and a correlation of point nine.
- The squares add to thirty six plus twenty five, sixty one.
- The correlation term is two times point nine times six times five, fifty four.
- Relative V a R is the square root of sixty one minus fifty four, root seven, about two point six five million.
- Subtracting the V a Rs gives one million, too low.
- Ignoring correlation gives root sixty one, about seven point eight one million, too high.
- Relative V a R is the natural basis for active risk limits.

26 Mapping derivatives

Decompose each derivative into positions in the risk factors

Instrument	Mapped as
Option	Delta × underlying; misses gamma
FX forward, buy foreign	Long spot and foreign zero; short domestic zero
FRA 6×12, pay fixed	Long 6-month zero, short 12-month zero
Swap, pay fixed	Short fixed bond, long floating-rate note
Swap, receive fixed	The reverse

IN THE LESSON

- Each derivative is mapped by decomposing it into positions in the risk factors.
- An option becomes delta times the underlying position, the delta normal approach, which misses gamma.
- A forward to buy foreign currency is long the foreign currency spot, long a foreign zero coupon bond, and short a domestic zero.
- A forward rate agreement, or F R A, fixes a rate for a future period.
- A six by twelve F R A that pays fixed is long a six month zero and short a twelve month zero of equal present value.
- A pay fixed swap is short a fixed rate bond and long a floating rate note.
- A receive fixed swap is the reverse.
- The sign on each leg decides whether risks offset.

27 Example 13: an option, delta and gamma

Calls on $20m of index: delta 0.6, dollar gamma ($\Gamma \times S^2 \times$ units) $40m. Index σ 5% a month

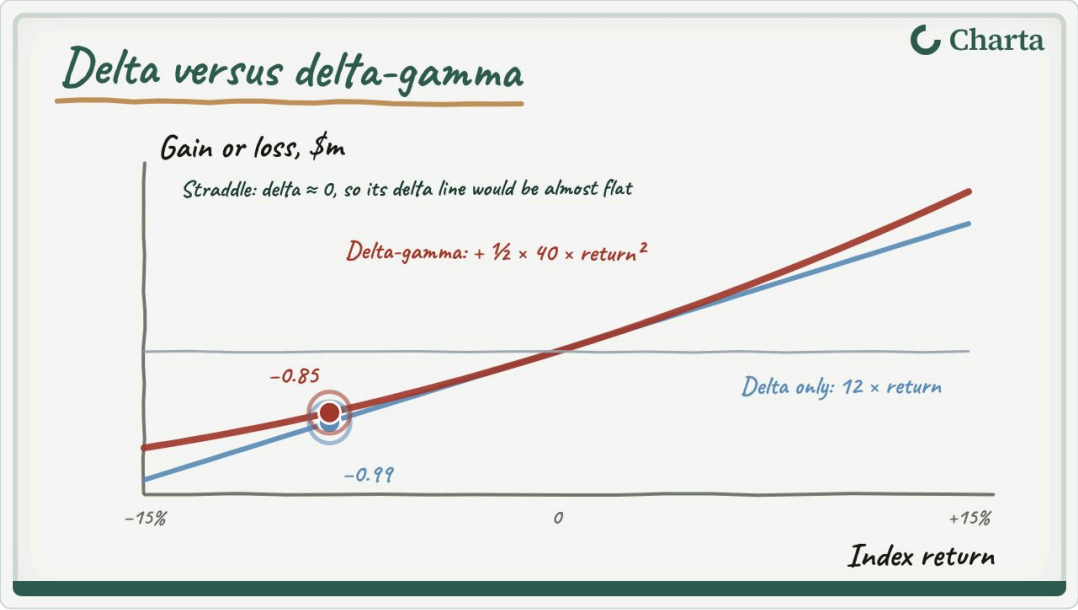
Delta exposure	0.6 × $20m	**$12m**
Delta-normal VaR	1.65 × 5% × $12m	**$990,000**
The 95% index move	−1.65 × 5%	**−8.25%**
Gamma gain	½ × 40m × 0.0825^2	**+$136,100**
Delta-gamma loss	990,000 − 136,100	**$853,900**

Long gamma: delta alone overstates the loss

IN THE LESSON

- Halden holds calls on twenty million dollars of the index, with delta point six and dollar gamma forty million.
- Dollar gamma is gamma times the index level squared times the units held.
- The delta exposure is point six times twenty million, twelve million dollars.
- Delta normal V a R is one point six five times five per cent times twelve million, nine hundred and ninety thousand dollars.
- That matches the ninety five per cent index move of minus one point six five times five per cent, minus eight point two five per cent.
- The gamma gain, one half times forty million times point zero eight two five squared, is about one hundred and thirty six thousand one hundred.
- So the delta gamma loss is about eight hundred and fifty three thousand nine hundred.
- Long gamma: delta alone overstates the loss.

28 Delta versus delta-gamma



IN THE LESSON

- Across the bottom is the month's index return, from minus fifteen to plus fifteen per cent.
- Up the side is the option's gain or loss, in millions of dollars.
- The straight line is the delta only mapping, twelve million times the return.
- The curve is the delta gamma view, adding one half times forty million times the return squared.
- They touch at a zero return, so small moves agree.
- At the minus eight point two five per cent move, the line shows a loss of point nine nine million, and the curve a loss of only point eight five million.
- And a straddle has a delta near zero, so its delta line would be almost flat: delta normal V a R would see almost no risk.

29 Example 14: a currency forward

Buy €10m in 1 year. Spot $1.10 per €. 1-year rates: € 3%, $ 5%

Forward rate	1.10 × 1.05 ÷ 1.03	**$1.1214 per €**
Long € 1-year zero	€10m ÷ 1.03 × 1.10	**+$10.68m**
Long € spot exposure	same € leg, FX risk factor	**+$10.68m**
Short $ 1-year zero	$11.214m ÷ 1.05	**−$10.68m**
Value at the start	€ leg 10.68 − $ leg 10.68	**$0**

Three risks remain: the exchange rate, € rates and $ rates

IN THE LESSON

- Halden buys ten million euros forward for one year.
- The spot rate is one point one zero dollars per euro, and one year rates are three per cent in euros and five per cent in dollars.
- The forward rate is one point one zero times one point zero five, over one point zero three, about one point one two one four dollars.
- Halden is long a one year euro zero, worth ten million euros over one point zero three, times spot: about ten point six eight million dollars.
- Through the F X risk factor, that same euro leg is also long the euro spot rate.
- And Halden is short a one year dollar zero paying eleven point two one four million, discounted at five per cent: also ten point six eight million.
- The value starts at zero, the euro leg minus the dollar leg, but three risks remain: the exchange rate, euro rates and dollar rates.

30 Example 15: a forward rate agreement

Pay-fixed 6×12 FRA on $10m. Simple rates: 6 months 4%, 12 months 4.5%

Long 6-month zero	10m ÷ (1 + 4% × 0.5)	**+$9.804m**
Forward rate, 6 to 12 months	2 × (1.045 ÷ 1.02 − 1)	**4.90%**
12-month zero pays	10m × (1 + 4.90% × 0.5)	**$10.245m**
Short 12-month zero, PV	10.245m ÷ 1.045	**−$9.804m**
If rates rise	the 12-month zero falls more	**FRA gains**

IN THE LESSON

- Halden enters a pay fixed six by twelve F R A on ten million dollars.
- Simple rates are four per cent for six months and four and a half for twelve.
- The long six month zero, ten million over one plus four per cent times one half, is worth about nine point eight zero four million.
- The implied forward rate is two times one point zero four five over one point zero two, minus one, about four point nine zero per cent.
- The twelve month zero pays ten million times one plus four point nine zero per cent times one half, about ten point two four five million.
- Discounted at four and a half per cent, the short twelve month zero is worth minus nine point eight zero four million.
- If rates rise, the twelve month zero falls more, and the F R A gains.

31 Example 16: an interest rate swap

Pay 5% fixed on $10m for 3 years; receive 1-year floating, just reset at 4%. $000

Short fixed bond (Example 4)	−481, −458, −9,070	**−10,009**
Long FRN: 10,400 in 1 year	10,400 ÷ 1.04	**+10,000**
Swap value	10,000 − 10,009	**−9**
Bond + swap: 1 year	481 − 481 + 10,000	**+10,000**
Bond + swap: 2 and 3 years	458 − 458; 9,070 − 9,070	**0; 0**

Pay fixed gains when rates rise: the fixed bond it is short falls

IN THE LESSON

- Halden pays a fixed five per cent on ten million dollars for three years and receives one year floating, which has just reset at four per cent.
- In thousands, the fixed leg is a short fixed bond: the bond from example four with the sign reversed, minus four eighty one, four fifty eight and nine thousand and seventy, minus ten thousand and nine in all.
- The long floating rate note has rate risk only to its next reset: it pays ten thousand four hundred in a year, worth ten thousand today.
- So the swap is worth about minus nine.
- Add Halden's long bond.
- At one year, four eighty one minus four eighty one plus ten thousand leaves ten thousand.
- At two and three years, the fixed legs cancel to zero.
- Pay fixed gains when rates rise, because the fixed bond it is short falls.

32 Mistakes to avoid

- Map first, then measure: never position by position
- Never add general and specific VaR: √ of the squares
- Net signed exposures at each vertex, then apply ρ
- Relative VaR is not VaR_P − VaR_B: use TEV or ρ
- Delta misses gamma; check the sign on every leg

IN THE LESSON

- Five mistakes cost marks.
- First, map first, then measure: never estimate risk position by position.
- Second, never add general and specific V a R.
- They are assumed uncorrelated, so take the square root of the sum of their squares.
- Third, net the signed exposures at each vertex, then apply the correlations.
- Fourth, relative V a R is not the portfolio V a R minus the benchmark V a R: use the tracking error volatility, or the formula with the correlation.
- And fifth, a delta mapping misses gamma, so check options with delta gamma, and check the sign on every leg of a forward, an F R A or a swap.

33 Try three quick questions

1. $2m stock, β 1.5, index σ 4%, residual σ 10%, z = 1.65. Total VaR?
   Answer: GVaR $198,000, SVaR $330,000; √ of squares ≈ $384,800.
2. Equal values; VaR_P $4m, VaR_B $3m, ρ = 0.5. Relative VaR?
   Answer: √(16 + 9 − 12) = √13 ≈ $3.61m.
3. How is a pay-fixed 3×9 FRA mapped?
   Answer: Long a 3-month zero, short a 9-month zero, equal PV.

IN THE LESSON

- Pause, think, then listen.
- Question one.
- A two million dollar stock has a beta of one point five, the index volatility is four per cent, the residual volatility is ten per cent, and z is one point six five.
- What is the total V a R?
- General V a R is one hundred and ninety eight thousand dollars and specific V a R three hundred and thirty thousand, so the total is about three hundred and eighty four thousand eight hundred.
- Question two.
- A portfolio and benchmark of equal value have V a Rs of four million and three million and a correlation of point five.
- Relative V a R?
- The root of sixteen plus nine minus twelve, about three point six one million.
- Question three.
- How is a pay fixed three by nine F R A mapped?
- Long a three month zero, short a nine month zero, of equal present value.