CHARTA LEARNING · LESSON NOTES

The Vasicek and Gauss+ Models

FRM Part II

1 A story: the hedge that broke

- Kestrel owns a 10-year bond, hedged by selling a 2-year
- One-factor model: both yields always move the same way
- Then the curve steepens: short rates fall, long rates rise
- Both legs lose; the model said that could not happen
- Lesson: slope moves need more than one factor

IN THE LESSON

- Kestrel Bank's rates desk owns a ten year government bond, and it hedges that bond by selling a two year bond.
- The desk's model has a single source of risk.
- In that model, whenever the two year yield moves, the ten year yield moves the same way, so the hedge looks perfect.
- Then the curve steepens: short rates fall while long rates rise.
- The desk loses on both legs.
- The bond it owns falls in price, and the bond it sold rises in price.
- Its model said that could not happen.
- The lesson is that a model with one factor cannot change the slope of the curve independently of its level.
- The desk needs a model with more than one factor.

2 The language of short-rate models

- Term structure model: how rates evolve, to price bonds
- Short rate r: the rate over a very short period
- Change = drift (predictable) + shock (random)
- Shock: σ × dW, a normal increment with variance dt
- Mean reversion: pulled toward a target at speed k or α

IN THE LESSON

- First, a few words we need.
- A term structure model describes how interest rates evolve over time, so that bonds of every maturity can be priced.
- The short rate, written r, is the rate for borrowing over a very short period.
- Its change over a tiny interval of time, d t, has two parts.
- The drift is the predictable part, the direction the rate is expected to move.
- The shock is the random part: sigma, the volatility, times d W, a random normal increment whose variance equals d t.
- Mean reversion means the drift pulls the rate back toward a target level, and the speed, written k in Vasicek and alpha in Gauss plus, says how hard it pulls.
- A larger k means a faster pull.

3 Gauss+: three factors in a cascade

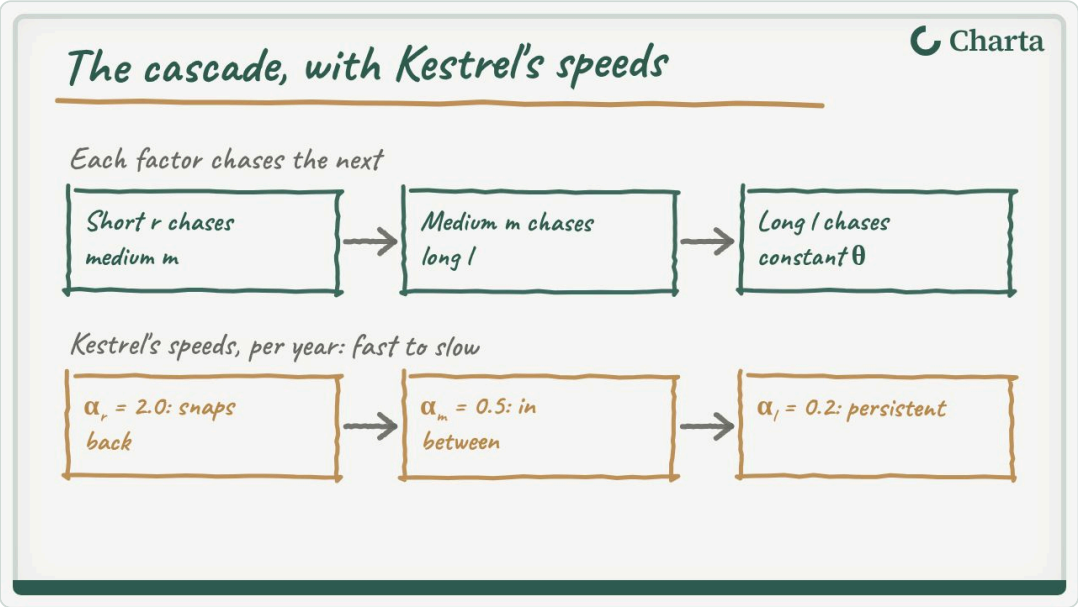
dr = −α_r(r − m)dt
dm = −α_m(m − l)dt + σ_m dW_m
dl = −α_l(l − θ)dt + σ_l dW_l
α_r > α_m > α_l (dW_m, dW_l may be correlated)

Short rate r	No shock: follows m
Medium rate m	Own shock; chases l
Long rate l	Own shock; chases θ

IN THE LESSON

- The Gauss plus model describes the term structure with three rate factors: a short rate r, a medium term rate m, and a long term rate l, linked as a cascade of mean reversion.
- Read the first line.
- The short rate's drift is minus its speed, alpha r, times the gap between the short rate and the medium rate, so the short rate is pulled toward m.
- It has no shock of its own: it simply follows the medium factor.
- The second line pulls the medium rate toward the long rate, at speed alpha m, and adds the medium factor's own shock.
- The third pulls the long rate toward a constant level, theta, at speed alpha l, with its own shock.
- The last line: the speeds fall down the cascade, and the two shocks may be correlated.
- In the boxes: r follows m with no shock, m chases l, and only l chases a constant.

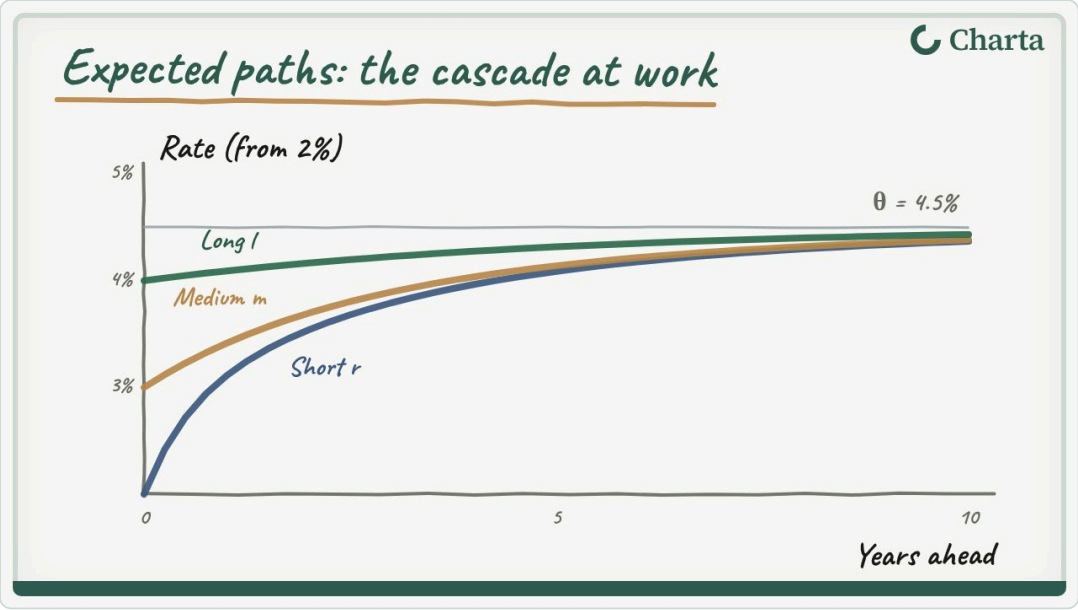
4 The cascade, with Kestrel's speeds



IN THE LESSON

- Here is the cascade as a picture.
- In the top row, each factor chases the next.
- The short rate chases the medium rate.
- The medium rate chases the long rate.
- And the long rate chases the constant, theta.
- So the short rate's target is not fixed.
- It moves, because m is itself drifting toward l, and l toward theta, and both are hit by shocks.
- In the bottom row are the speeds Kestrel's desk has chosen, all per year.
- The short rate has a speed of two, so it snaps back quickly to wherever m is.
- The medium rate has a speed of point five, in between.
- The long rate has a speed of point two, so it is persistent: a shock to it fades only slowly, over many years.
- Fast at the front, slow at the back.
- That ordering is the heart of the model.

5 Expected paths: the cascade at work



IN THE LESSON

- Here is the cascade at work, ignoring the shocks so that we see only the drift.
- Along the bottom is time in years, from zero to ten.
- Up the side is the rate, from two per cent at the bottom to five per cent.
- Kestrel starts today with the short rate at two per cent, the medium rate at three and the long rate at four.
- Theta, the grey line, is four point five per cent.
- The blue short rate rises steeply at first, closing most of its gap to the medium rate within about a year.
- The gold medium rate climbs more gently toward the long rate.
- The green long rate creeps up toward theta.
- After that, the short and medium rates ride along with the slow long rate, and all three get close to theta only after many years.

6 Why three factors: level, slope and curvature

- Three factors, two shocks (on m and l); r lags m
- Level: all rates move up or down together
- Slope: the long–short gap widens or narrows
- Curvature: the middle bends; r's lag behind m adds to it
- Trap: a chain, not three rates reverting to constants

IN THE LESSON

- Why three factors?
- A one factor model has a single shock, so it can move the curve in only one fixed pattern.
- Gauss plus has three factors but only two random shocks, on the medium and long factors, and the short rate lags behind the medium factor at its own speed.
- So the curve's level, slope and curvature are no longer locked together.
- Its level, with all rates moving up or down together.
- Its slope, with the gap between long and short rates widening or narrowing, because the two shocks can push the medium and long factors apart.
- And its curvature, with the middle of the curve bending relative to the two ends, shaped further by the short rate's lag behind the medium factor.
- The steepening that broke Kestrel's hedge is now possible, because the medium factor, and the short rate behind it, can fall while the long factor rises.
- The common mistake is to treat the three as independent rates, each reverting to its own constant.
- They are not.
- They are a chain.

7 Example 1: chase the chain, not constants

Kestrel: r 2.00%, m 3.00%, l 4.00%, θ 4.50%. Drift over one quarter, dt = 0.25

Short drift, toward m	2.0 × (3.00 − 2.00) × 0.25	**+0.50%**
Medium drift, toward l	0.5 × (4.00 − 3.00) × 0.25	**+0.125%**
Long drift, toward θ	0.2 × (4.50 − 4.00) × 0.25	**+0.025%**
Wrong: short toward θ	2.0 × (4.50 − 2.00) × 0.25	**+1.25%**

IN THE LESSON

- Kestrel's levels are as before: short two per cent, medium three, long four, and theta four point five.
- Take one quarter of a year, so d t is point two five, and look only at the drift.
- The short rate's drift is its speed of two, times the gap from two up to three, which is one percentage point, times point two five: plus point five zero per cent.
- The medium rate's drift is point five times one, times point two five: plus point one two five per cent.
- The long rate's drift is point two times point five, times point two five: plus point zero two five per cent.
- Now the trap.
- If you send the short rate straight to theta, the gap is two point five, and the drift becomes plus one point two five per cent, two and a half times too large.

8 Vasicek: one factor

dr = k(θ − r)dt + σ dW
One shock, dW, drives every rate on the curve

Speed k	Speed of reversion
Level θ	Constant long-run level
Volatility σ	Constant volatility
Shock dW	The only source of risk

IN THE LESSON

- Now the model that Gauss plus is compared with.
- The Vasicek model is a one factor model.
- The short rate follows d r equals k times theta minus r, d t, plus sigma d W.
- It reverts at speed k toward one constant level, theta, with a constant volatility, sigma.
- The second line is the key fact.
- There is one shock, d W, and it drives every rate on the curve, because every bond price in the model depends only on today's short rate.
- In the boxes: k is the speed of reversion.
- Theta is a constant long run level, not a moving target.
- Sigma is a constant volatility.
- And d W is the only shock.
- Remember it, because the common mistake is to say that Vasicek has several sources of risk.
- It has one.

9 What one factor implies

- All yields perfectly correlated: one shock moves all
- Limited moves: no independent slope or curvature change
- Yield volatility falls steadily as maturity rises
- Why: mean reversion makes a shock to r fade
- Plus: closed-form bond prices; a teaching benchmark

IN THE LESSON

- What does a single factor imply?
- First, all yields are perfectly correlated.
- If the two year yield rises, every other yield rises too, by an amount the model fixes.
- Second, the curve moves in a limited way.
- It cannot produce independent changes in slope or curvature, and that is exactly what broke Kestrel's hedge.
- Third, the volatility of yields falls steadily as maturity rises.
- The reason is mean reversion: a shock to today's short rate is expected to fade long before a long bond matures.
- On the plus side, Vasicek gives closed form bond prices, meaning exact formulas with no simulation needed.
- That makes it a good teaching and benchmark model.

10 Example 2: one shock, every yield

Vasicek, k = 0.25, Δr = 100 bp. Δy(T) = (1 − e^(−kT)) ÷ (kT) × Δr

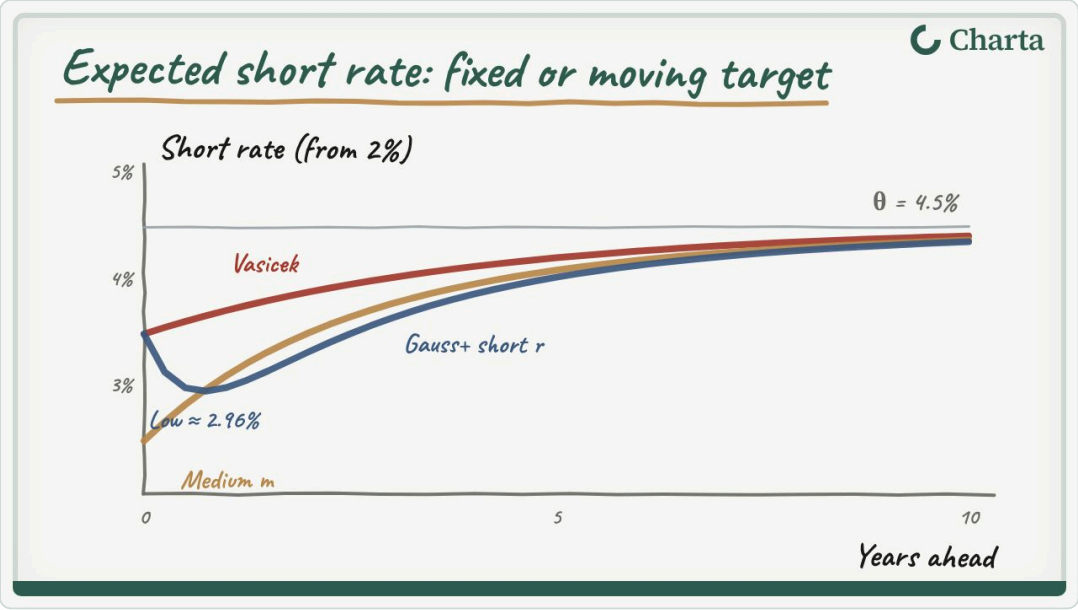
1-year yield	(1 − e^(−0.25)) ÷ 0.25 × 100	**+88 bp**
5-year yield	(1 − e^(−1.25)) ÷ 1.25 × 100	**+57 bp**
10-year yield	(1 − e^(−2.5)) ÷ 2.5 × 100	**+37 bp**
30-year yield	(1 − e^(−7.5)) ÷ 7.5 × 100	**+13 bp**

Every yield moves the same way; the size fades with maturity

IN THE LESSON

- Take a Vasicek model with a speed of point two five, and let the short rate jump by one percentage point, which is one hundred basis points.
- A basis point is one hundredth of a per cent.
- In Vasicek, a yield of maturity T moves by one minus e to the minus k T, all over k T, times the change in the short rate.
- So each multiplier is applied to one hundred basis points.
- The one year yield rises by about eighty eight basis points.
- The five year yield by about fifty seven.
- The ten year by about thirty seven.
- And the thirty year by about thirteen.
- Every yield moves up together, and the move fades as maturity rises.
- In this model there is no way to make the ten year yield rise while the two year yield falls.

11 Expected short rate: fixed or moving target



IN THE LESSON

- Now compare where each model expects the short rate to go.
- Suppose the short rate is three point five per cent, but the medium factor is lower, at two point five, and the long factor is at four.
- Theta, the grey line, is four point five.
- Along the bottom is time in years, from zero to ten, and up the side the rate, from two to five per cent.
- The red line is Vasicek, with a speed of point two five.
- It rises smoothly toward theta, because its target is a constant.
- The blue line is the Gauss plus short rate, and the gold line is the medium factor, starting at two point five per cent.
- The short rate first falls, to a low of about two point nine six per cent after about nine months, where it meets the medium factor.
- Then it turns and rises with the medium factor as it climbs.
- A fixed target gives a smooth path.
- A moving target can give a dip.

12 Example 3: the first quarter's drift

r 3.50%, m 2.50%, l 4.00%, θ 4.50%. Vasicek k = 0.25. One quarter, dt = 0.25

Vasicek short drift	0.25 × (4.50 − 3.50) × 0.25	**+0.0625%**
Gauss+ short drift	2.0 × (2.50 − 3.50) × 0.25	**−0.50%**
Gauss+ medium drift	0.5 × (4.00 − 2.50) × 0.25	**+0.1875%**
Gauss+ long drift	0.2 × (4.50 − 4.00) × 0.25	**+0.025%**

IN THE LESSON

- Here are the first quarter's drifts behind that picture.
- The short rate is three point five per cent, the medium factor two point five, the long factor four, and theta four point five.
- Vasicek's speed is point two five, and the step is one quarter, so d t is point two five.
- Vasicek's drift is point two five, times the gap of one point up to theta, times point two five: plus point zero six two five per cent, a small rise.
- The Gauss plus short drift is two, times the gap of minus one point down to the medium factor, times point two five: minus point five zero per cent.
- Meanwhile, the medium factor's drift is point five times one point five, times point two five: plus point one eight seven five per cent.
- The long factor's drift is plus point zero two five.
- The short rate falls now while its target rises, and that is the dip.

13 Vasicek versus Gauss+

Dynamics, characteristics and uses

	Vasicek	Gauss+
Factors	One: r	Three: r, m, l
Short rate reverts to	Constant θ	Moving target m
Yields	Perfectly correlated	Not perfectly
Curve moves	One pattern	Level, slope, curvature
Best use	Teaching, benchmark	Curve risk, hedging

IN THE LESSON

- Let us put the two side by side: dynamics, characteristics and uses, row by row.
- Factors: Vasicek has one, the short rate.
- Gauss plus has three, the short, medium and long factors.
- What the short rate reverts to: in Vasicek a constant, theta.
- In Gauss plus a moving target, the medium factor.
- Yields: in Vasicek perfectly correlated.
- In Gauss plus not perfectly correlated, because two shocks, to the medium and long factors, drive them.
- Curve moves: Vasicek produces only one pattern, while Gauss plus, with its two shocks and the short rate's lag, can change the level, the slope and the curvature, so it reproduces non parallel moves.
- And the best use: Vasicek, with its closed form prices, is a teaching tool and a benchmark.
- Gauss plus suits measuring curve risk and hedging it.

14 Kestrel's choice, and two traps

- Vasicek: one factor, so one hedging bond
- Gauss+: shocks to m and l, so one hedge is not enough
- Hedge with bonds at different points on the curve
- Trap: Vasicek has one source of risk, not several
- Trap: the Gauss+ short rate reverts to m, not a constant

IN THE LESSON

- Back to Kestrel's desk.
- In Vasicek, one factor means one hedge.
- Any bond can hedge any other, since both respond to the same shock, and that is why selling the two year looked like a perfect hedge for the ten year.
- In Gauss plus, a position is exposed to more than one shock, the medium and the long, so a single hedging bond is no longer enough.
- The desk needs several hedges, for example bonds at different points on the curve.
- That is why Gauss plus suits curve risk and hedging.
- Keep the two traps of this comparison in mind.
- Vasicek has one source of risk, never several.
- And in Gauss plus, the short rate reverts to the medium factor, not to a constant.

15 Moving a factor over a short step

Move = α × (target − level) × dt + σ × √dt × z
Short rate: drift only, no shock term
z: a standard normal draw

Drift term	Proportional to dt
Shock term	Scales with √dt; m and l only
Target	m for r; l for m; θ for l

IN THE LESSON

- Now the calculation.
- To move a Gauss plus factor over a short interval, write its equation in discrete form.
- The move equals the speed alpha, times the target minus the level, times d t, plus sigma times the square root of d t, times z.
- Here z is a standard normal draw, a random number with mean zero and standard deviation one.
- The second line matters: the short rate has no shock term, so its move is the drift alone.
- Now the boxes.
- The first term is the drift, which is proportional to d t.
- The second is the shock, which scales with the square root of d t, because the variance of the shock grows in line with time.
- The target depends on the factor: the medium rate for the short factor, the long rate for the medium factor, and theta for the long factor.

16 Rules before you compute

- Target: m for r, l for m, θ for l
- Gap = target − level: below target drifts up
- Short rate: no shock; only m and l are shocked
- Start-of-step levels: never reuse an updated factor
- dt in years: a quarter is 0.25, a month is 1/12

IN THE LESSON

- A few rules keep the calculation straight.
- First, choose the right target for each factor: m for the short rate, l for the medium rate, theta for the long rate.
- Second, the gap is the target minus the level, in that order, so a factor below its target drifts up and a factor above it drifts down.
- Third, the short rate gets no shock: only the medium and long factors are shocked.
- Fourth, all three moves use the levels at the start of the step, so never update one factor and then reuse its new value within the same step.
- Finally, measure d t in years, so a quarter is point two five, and a month is one twelfth.
- Each rule matches a typical exam error.

17 Example 4: the shocks for one quarter

σ_m 0.80%, σ_l 0.50%. Draws z: −0.5 for m, +0.4 for l. dt = 0.25

Square root of dt	√0.25	**0.5**
Short shock	none: r has no shock	**0**
Medium shock	0.80 × 0.5 × (−0.5)	**−0.20%**
Long shock	0.50 × 0.5 × (+0.4)	**+0.10%**

IN THE LESSON

- Kestrel's volatilities are point eight zero per cent for the medium factor and point five zero per cent for the long one.
- The random draws are minus point five for the medium factor and plus point four for the long factor, and d t is point two five.
- In general the two draws may be correlated; here they are simply given.
- We start from the levels of example one, two, three and four per cent, whose drifts we already have.
- The square root of d t is point five.
- The short rate gets no shock at all, so its shock is zero.
- The medium shock is point eight zero times point five times minus point five, minus point two zero per cent.
- The long shock is point five zero times point five times point four, plus point one zero per cent.

18 Example 4: the moves and the new levels

Move = drift + shock, from start levels r 2.00%, m 3.00%, l 4.00%

Short: move, new level	drift only: +0.50	**2.50%**
Medium: move, new level	+0.125 − 0.20 = −0.075	**2.925%**
Long: move, new level	+0.025 + 0.10 = +0.125	**4.125%**
Slope l − r: before, after	4.00 − 2.00; 4.125 − 2.50	**2.00 → 1.625**

IN THE LESSON

- Each move is the drift plus the shock, applied to the levels at the start: two, three and four per cent.
- The short factor moves by its drift alone, plus point five zero, so it ends at two point five zero per cent.
- The medium factor moves by plus point one two five minus point two zero, a fall of point zero seven five, to two point nine two five per cent.
- Its drift was up, but the shock pulled it down.
- The long factor moves by plus point zero two five plus point one zero, plus point one two five, to four point one two five per cent.
- The slope, long minus short, was two percentage points and is now one point six two five.
- In one quarter the curve has flattened, because its ends moved by different amounts.
- Next quarter, the short rate will chase the lower medium factor.

19 Example 5: three errors, three wrong answers

Correct moves this quarter: short +0.50%, medium −0.075%

m: shock with dt, not √dt	0.125 + 0.80 × 0.25 × (−0.5)	**+0.025%**
r: gap sign flipped	2.0 × (2.00 − 3.00) × 0.25	**−0.50%**
r: sequential, new m 2.925	2.0 × 0.925 × 0.25	**+0.4625%**
Correct: start-of-step levels	r drift; m drift + shock	**+0.50%, −0.075%**

IN THE LESSON

- Now the three typical errors.
- The correct moves this quarter are plus point five zero per cent for the short factor and minus point zero seven five for the medium factor.
- First, using d t in place of its square root for the medium shock.
- The shock becomes point eight zero times point two five times minus point five, only minus point one zero, and the medium move is plus point zero two five: the wrong sign.
- Second, flipping the sign of the short rate's gap.
- The drift becomes minus point five zero, the wrong direction altogether.
- Third, updating sequentially: moving the medium factor first, then using its new level, two point nine two five, as the short rate's target.
- The gap is point nine two five, and the short move is plus point four six two five.
- The correct answers use start of step levels.

20 Example 6: a much shorter step

Same medium factor and draw z = −0.5, but dt = 0.01 of a year

Square root of dt	√0.01	**0.1**
Drift	0.5 × 1.00 × 0.01	**+0.005%**
Shock	0.80 × 0.1 × (−0.5)	**−0.04%**
Move	0.005 − 0.04	**−0.035%**

dt 25 times smaller: drift 25 times smaller, shock only 5 times

IN THE LESSON

- What happens over a much shorter step?
- Keep Kestrel's medium factor and the same draw of minus point five, but let d t be one hundredth of a year, roughly two and a half trading days.
- The square root of d t is point one.
- The drift is point five times the gap of one, times point zero one: plus point zero zero five per cent, half a basis point.
- The shock is point eight zero times point one times minus point five: minus point zero four per cent, four basis points.
- The move is minus point zero three five per cent.
- Compared with the quarter, d t is twenty five times smaller, so the drift is twenty five times smaller, from point one two five to point zero zero five.
- The shock is only five times smaller, from minus point two zero to minus point zero four.
- Over short steps, randomness dominates.

21 Estimating Gauss+ from history

- Data: a time series of short, medium and long rates
- Gap form: regress the change on (target − level)
- Lagged form: change on the lagged level, plus an intercept
- Coefficients give speeds; residual spread gives σ for m, l
- Last step: convert them to annual terms

IN THE LESSON

- Where do the speeds, levels and volatilities come from?
- They can be estimated from history.
- Collect a time series of short, medium and long rates, observed at regular intervals, say every quarter.
- For each factor, regress its change over one period on the gap between its target and its level.
- That is the gap form.
- Or regress the change on the factor's own lagged level, with an intercept.
- That is the lagged level form, and it suits the long factor, whose target is the unknown constant theta.
- The fitted coefficients give the speeds per period.
- The spread of the residuals, the parts the regression cannot explain, gives the volatility per period, for the medium and long factors only, since the short rate has no shock of its own.
- The last step is to convert them to annual terms.

22 From regression output to annual parameters

Gap form: Δx = slope × (target − x) + e, so α = slope ÷ dt
Lagged form: Δx = a + b x_{t−1} + e, so α = −b ÷ dt
Level = −a ÷ b
σ = sd(e) ÷ √dt (m and l only)

Speed	Slope = α × dt: divide by dt
Long-run level	−a ÷ b: where change is zero
Volatility	sd(e) = σ√dt: divide by √dt

IN THE LESSON

- Here are the conversions.
- In the gap form, the change in a factor x equals a slope times the gap, plus a residual, e.
- Comparing this with the model, the slope equals alpha times d t, so the annual speed is the slope divided by d t.
- In the lagged level form, the change equals an intercept, ay, plus b times last period's level, plus a residual.
- Then the speed is minus b divided by d t, and the long run level, where the expected change is zero, is minus ay divided by b.
- The last line: for the medium and long factors, the residual standard deviation estimates sigma times the square root of d t, so the annual volatility is the residual standard deviation divided by the square root of d t.
- The boxes sum it up: speed, level and volatility, each converted to annual terms.

23 Example 7: the short factor from quarterly data

Kestrel's quarterly history, dt = 0.25. One data point, then the fitted output

One point: gap at quarter t	m − r = 2.90 − 2.10	**0.80%**
Its change to quarter t+1	r: 2.10 → 2.50	**+0.40%**
Fitted slope, all quarters	regression output	**0.50**
Speed α_r	0.50 ÷ 0.25	**2.0**

No σ_r to estimate: the short rate has no shock of its own

IN THE LESSON

- Kestrel checks the parameters it assumed, using its own quarterly history, so d t is point two five.
- Each quarter gives one data point.
- For example, in one quarter the medium rate is two point nine zero per cent and the short rate two point one zero, a gap of point eight zero.
- By the next quarter the short rate has risen from two point one zero to two point five zero, a change of plus point four zero.
- That pair, the gap and the change, is one point in the regression.
- This point sits on the fitted line: across all quarters, the fitted slope is point five zero, and point five zero times point eight zero is point four zero.
- The speed is point five zero divided by point two five, which is two, matching what the desk assumed.
- There is no short rate volatility to estimate, because in Gauss plus the short rate has no shock of its own.

24 Example 8: the medium and long factors

dt = 0.25. Long factor: Δl = 0.225 − 0.05 × l_{t−1} + e

Medium speed α_m	slope 0.125 ÷ 0.25	**0.5**
Medium volatility σ_m	sd 0.40 ÷ √0.25	**0.80%**
Long speed α_l	−b ÷ dt = 0.05 ÷ 0.25	**0.2**
Long-run level θ	−a ÷ b = 0.225 ÷ 0.05	**4.50%**
Long volatility σ_l	sd 0.25 ÷ √0.25	**0.50%**

IN THE LESSON

- Both regressions use quarterly data, so d t is still point two five.
- The long factor's regression on its lagged level came out as: change equals point two two five, minus point zero five times last quarter's level.
- For the medium factor, in gap form, the fitted slope is point one two five, so the speed is point one two five divided by point two five, which is point five.
- Its residual standard deviation is point four zero, so its volatility is point four zero divided by point five, point eight zero per cent.
- For the long factor, b is minus point zero five, so the speed is point zero five divided by point two five, which is point two.
- The long run level is minus ay over b, point two two five divided by point zero five: four point five zero per cent.
- Its residual standard deviation of point two five, divided by point five, gives a volatility of point five zero per cent.

25 Example 9: the long factor is imprecise

Standard errors: short slope 0.05; long b 0.04. Range ≈ estimate ± 2 s.e.

Short slope range	0.50 ± 0.10	**0.40 to 0.60**
Short speed range	÷ 0.25	**1.6 to 2.4**
Long b range	−0.05 ± 0.08	**−0.13 to +0.03**
Long speed range	−b ÷ 0.25	**−0.12 to 0.52**

Slow factors, especially the long one, are estimated imprecisely

IN THE LESSON

- Now the exam caveat.
- Every regression estimate comes with a standard error, and a rough ninety five per cent range is the estimate plus or minus two standard errors.
- Suppose the short factor's slope has a standard error of point zero five, and the long factor's b has one of point zero four.
- The short slope lies between point four zero and point six zero, so dividing by point two five, the speed lies between one point six and two point four: reasonably pinned down.
- The long factor's b lies between minus point one three and plus point zero three, so its speed lies between minus point one two and point five two.
- That range includes zero, which means no mean reversion at all, and then theta could be almost anything.
- A slow factor returns to its level only a few times in a sample, so the data say little about it.

26 Mistakes to avoid

- Cascade: r to m (no shock), m to l, l to θ
- Vasicek has one source of risk, not several
- Shock uses √dt; the gap is target − level
- Move all three factors from start-of-step levels
- Annualise: α = slope ÷ dt; σ = sd ÷ √dt

IN THE LESSON

- Here are the mistakes to avoid across the whole topic.
- Gauss plus is a cascade: the short rate reverts to the medium factor, with no shock of its own, the medium factor to the long factor, and only the long factor to a constant.
- Vasicek has one source of risk, not several, so its yields are perfectly correlated.
- When you move a factor, the shock uses the square root of d t, and the gap is the target minus the level.
- Move all three factors from the levels at the start of the step.
- And when you estimate, convert to annual terms: the speed is the slope over d t, and the volatility is the residual standard deviation over the square root of d t.

27 Recap: the main ideas

- Gauss+: three factors in a cascade of mean reversion
- α_r > α_m > α_l (fast to slow); r has no shock; m and l do
- Vasicek: one factor, constant θ, closed-form prices
- Move = α × gap × dt + σ × √dt × z (no σ term for r)
- Estimate by regression; annualise; long factor imprecise

IN THE LESSON

- Let us pull it together.
- Gauss plus has three factors, short, medium and long, in a cascade of mean reversion.
- The speeds fall down the cascade, so the short rate snaps back to the medium factor and the long factor is persistent.
- Only the medium and long factors carry shocks, and those two shocks, with the short rate's lag behind the medium factor, let the curve change its level, slope and curvature.
- Vasicek has one factor reverting to a constant theta, so its yields are perfectly correlated, but it gives closed form prices and serves as a benchmark.
- Each factor moves by its speed times its gap times d t, plus, for the medium and long factors, sigma times the root of d t times a normal draw.
- And the parameters come from regressions on historical rates, converted to annual terms, with the long factor the least precise.

28 Try three quick questions

1. m 3.20%, l 3.70%, α_m 0.5, σ_m 1.0%, dt 0.04, z = −1. Move in m?
   Answer: 0.5 × 0.50 × 0.04 + 1.0 × 0.2 × (−1) = −0.19%.
2. Monthly: Δl = 0.012 − 0.003 × l_{t−1}. Speed and level?
   Answer: α_l = 0.003 × 12 = 0.036; level = 0.012 ÷ 0.003 = 4.0%.
3. How many sources of risk does Vasicek have?
   Answer: One: all its yields are perfectly correlated.

IN THE LESSON

- Now try three quick questions.
- Pause, think, then listen.
- Question one.
- In Gauss plus, the medium factor is three point two zero per cent and the long factor three point seven zero.
- The medium speed is point five, its volatility one per cent, d t is point zero four and the draw is minus one.
- What is the move in the medium factor?
- The drift is point five times point five zero times point zero four, plus point zero one.
- The shock is one times point two times minus one, minus point two zero.
- The move is minus point one nine per cent.
- Question two.
- Monthly data give: change in the long factor equals point zero one two, minus point zero zero three times its lagged level.
- What are the speed and the level?
- The speed is point zero zero three times twelve, point zero three six, and the level is four per cent.
- Question three.
- How many sources of risk does Vasicek have?
- One, so all its yields are perfectly correlated.