CHARTA LEARNING · LESSON NOTES

Volatility Smiles and Volatility Surfaces

FRM Part II

1 A story: one volatility for every strike

- Index at 100; analyst uses one vol, 20%, for every strike
- Her 80-strike put looks too cheap next to the market
- Her 115-strike call looks too dear
- The market says: volatility depends on the strike

IN THE LESSON

- Kestrel Bank's equity desk sells options on a stock index that stands at one hundred.
- A new analyst prices every six month option with a single volatility, twenty per cent.
- Her price for a put with a strike of eighty, far below today's level, looks too cheap next to what the market charges.
- Her price for a call with a strike of one hundred and fifteen looks too dear.
- The market is telling her that volatility depends on the strike.
- To read the message, we first need implied volatility.

2 Implied volatility

- BSM inputs: S0, K, T, r, dividend yield, volatility σ
- All but σ are observed, and the price is quoted
- Implied vol: the σ that makes BSM equal the market price
- Solve by trial: the price always rises with volatility
- Lognormal, constant vol: one σ for every strike

IN THE LESSON

- The Black Scholes Merton formula, which we shorten to B S M, prices a European option from a handful of inputs: today's asset price, S nought, the strike K, the time to expiry T, the risk free rate r, any dividend yield, and the volatility, sigma.
- All of them but sigma can be observed, and the option's price is quoted in the market.
- So we turn the formula round.
- Implied volatility is the volatility that, put into the formula, reproduces the option's market price.
- We solve by trial, because an option's price always rises with volatility.
- If the model's assumptions held, with lognormal prices and constant volatility, every option on one underlying with one maturity would give the same implied volatility, whatever its strike.

3 Example 1: finding an implied volatility

6-month index call, S0 = K = 100, r = 4%, no dividends. Market price $6.63

Try 18%	BSM price below market	**$6.08**
Try 22%	BSM price above market	**$7.18**
Try 20%	BSM price equals market	**$6.63**
Implied volatility	the σ that matches $6.63	**20%**

IN THE LESSON

- Here is the trial at work.
- Kestrel's six month index call has a strike of one hundred, the same as the index.
- The rate is four per cent and there are no dividends.
- The market price is six dollars and sixty three cents.
- First we try a volatility of eighteen per cent.
- The formula gives six dollars and eight cents, below the market, so the volatility must be higher.
- Next we try twenty two per cent, which gives seven dollars and eighteen cents, now above the market.
- Trying twenty per cent gives six dollars and sixty three cents, the market price exactly.
- The implied volatility of this call is therefore twenty per cent.

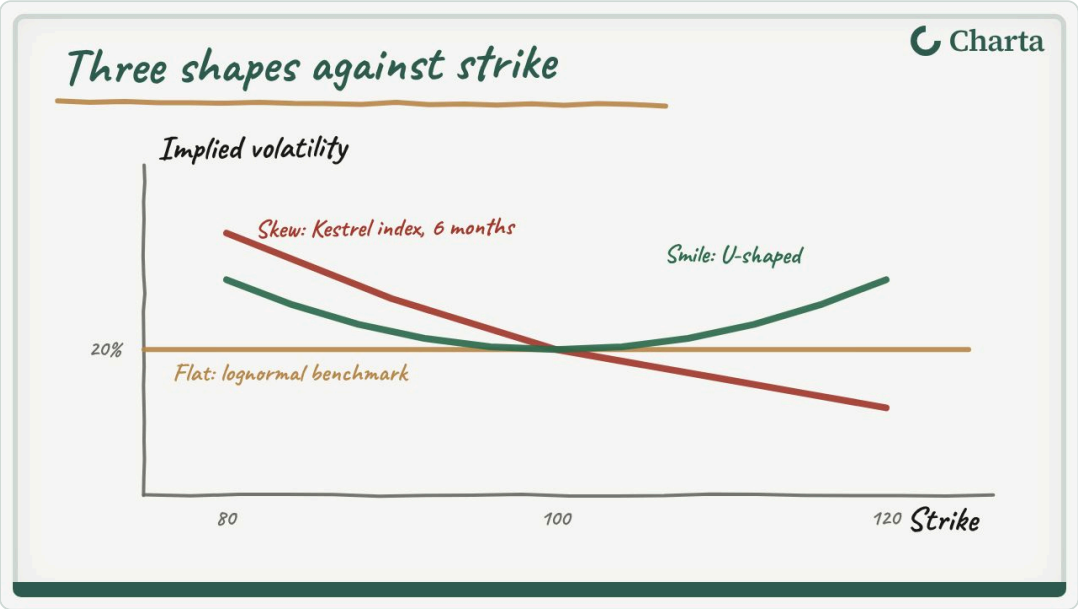
4 Smile and skew

- Implied vol against strike, one maturity: the smile
- Smile: U-shaped, higher at low and high strikes
- Skew or smirk: mostly downward, highest at low strikes
- Flat line: the market agrees with the lognormal
- Trap: the term structure varies maturity, not strike

IN THE LESSON

- Now repeat the search for every strike with the same maturity, and plot implied volatility against strike.
- That plot is called the volatility smile, whatever its shape.
- Two shapes matter.
- A smile in the narrow sense is U shaped, with higher implied volatility at both low and high strikes than near the money.
- A skew, also called a smirk, slopes mostly downward, with the highest implied volatility at the lowest strikes.
- A flat line would mean the market agrees with a lognormal distribution.
- One trap: the term structure of volatility varies the maturity, not the strike, so do not confuse it with either shape.

5 Three shapes against strike



IN THE LESSON

- Here are the three shapes on one picture.
- Across the bottom is the strike, from below eighty to above one hundred and twenty, and up the side is implied volatility.
- The gold line is flat at twenty per cent.
- The model's benchmark.
- The red line is Kestrel's own six month index skew.
- It starts at twenty eight per cent at a strike of eighty, passes twenty per cent at one hundred, and falls to sixteen per cent at one hundred and twenty.
- The green curve is a U shaped smile, the kind we meet in currencies.
- The labels name them: flat, the lognormal benchmark, then the skew, then the U shaped smile.

6 Put-call parity

c + K e^{−rT} = p + S0
Holds for market prices and for BSM at any one σ

European call price, c	Call, same strike and expiry
Strike discounted, K e^{−rT}	Cash that grows to the strike
European put price, p	Put, same strike and expiry
Asset price today, S0	The asset itself

IN THE LESSON

- Put call parity links a call and a put on the same asset with the same strike and expiry.
- Holding the call, plus cash equal to the strike discounted at the risk free rate, gives exactly the same payoff at expiry as holding the put plus the asset.
- So the two must cost the same today: c plus K times e to the minus r T equals p plus S nought.
- For a dividend or a foreign rate, a term is added for that income.
- The relation rests only on arbitrage, so it holds for market prices.
- It also holds for B S M prices computed with any single volatility.

7 Why a call and a put share one implied vol

- Parity holds for market prices: no arbitrage
- It also holds for BSM prices at any one volatility
- Subtract: the call's market minus model gap = the put's
- Both rise with σ, so one σ closes both gaps
- Same strike and expiry: one implied volatility

IN THE LESSON

- Follow the logic in four steps.
- First, parity holds for market prices.
- Second, it also holds for model prices at any one volatility.
- Third, subtract the model version from the market version.
- The discounted strike and the asset price cancel, and what is left says the call's gap between market price and model price equals the put's gap between market price and model price.
- Fourth, both prices rise with volatility, so the volatility that closes the call's gap closes the put's gap as well.
- So a European call and a European put with the same strike and expiry must have the same implied volatility.
- If they did not, there would be an arbitrage.

8 Example 2: parity in numbers

6-month index options, strike 90, S0 = 100, r = 4%. The put's implied vol is 23.5%

Discounted strike	90 × e^{−0.04 × 0.5}	**$88.22**
Market put	BSM at 23.5%	**$2.03**
Market call, from parity	2.03 + 100 − 88.22	**$13.81**
Put gap over BSM at 20%	2.03 − 1.37	**$0.66**
Call gap over BSM at 20%	13.81 − 13.15	**$0.66**

Equal gaps, so the call's implied vol is also 23.5%

IN THE LESSON

- Kestrel's six month index put with a strike of ninety trades at a price that implies twenty three point five per cent, the skew's own figure.
- The rate is four per cent.
- The discounted strike is ninety times e to the minus point zero two, eighty eight dollars and twenty two cents.
- The market put costs two dollars and three cents.
- Parity then fixes the market call: two oh three plus one hundred minus eighty eight twenty two, which is thirteen dollars and eighty one cents.
- Now compare with the model at twenty per cent.
- The model put is one dollar thirty seven, a gap of sixty six cents.
- The model call is thirteen dollars fifteen, also a gap of sixty six cents.
- The gaps match, so the call's implied volatility is twenty three point five per cent too.

9 What parity means in practice

- Build the smile from the more liquid OTM options
- Low strikes: OTM puts. High strikes: OTM calls
- Different call and put vols at one strike: arbitrage
- Parity is exact only for European options
- American exercise, dividend estimates: small differences

IN THE LESSON

- Three consequences follow.
- First, the smile can be built from whichever option is easier to trade.
- Out of the money options are usually the more liquid, so the smile is built from out of the money puts at low strikes and out of the money calls at high strikes.
- Second, if a quoted call and put at the same strike and expiry show different implied volatilities, that difference signals an arbitrage, or a stale quote.
- Third, parity is exact only for European options.
- American options, which can be exercised early, and errors in the dividend estimate can create small differences between call and put implied volatilities, and those are not arbitrages.

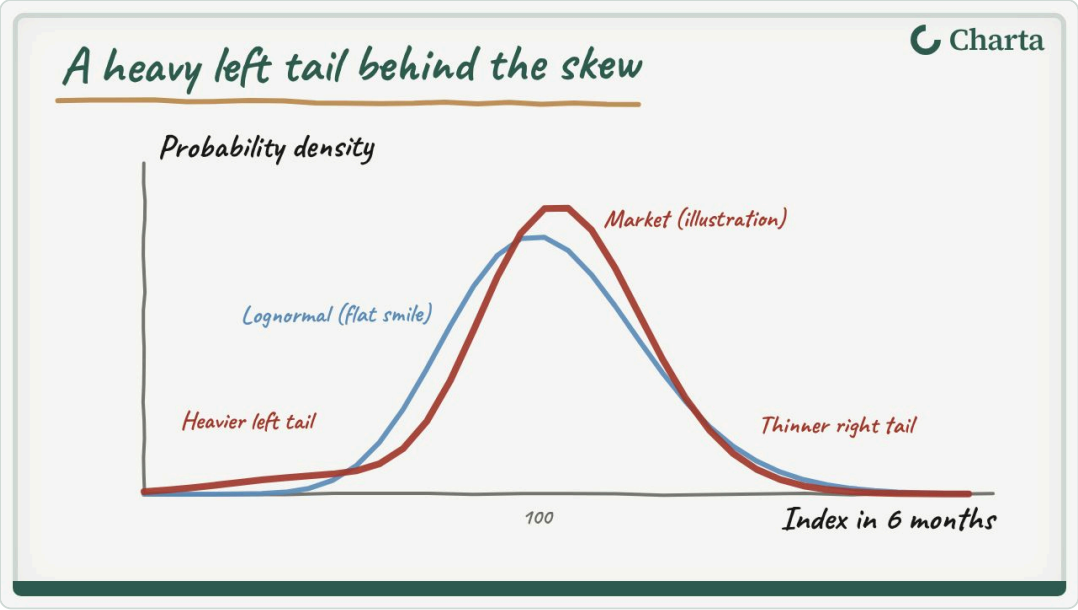
10 Two descriptions of one distribution

- Prices and vols: one risk-neutral distribution
- Flat smile: the lognormal distribution
- More probability in a region: options there cost more
- Rule: a heavy tail raises the implied vol there
- One at-the-money vol for every strike misprices the tails

IN THE LESSON

- Why does the smile take a particular shape?
- Option prices and implied volatilities are two descriptions of one thing: the risk neutral distribution of the asset price at maturity.
- A flat smile corresponds to the lognormal distribution that the model assumes.
- Where the market puts more probability than the lognormal on a region of prices, options struck in that region pay off more often.
- They cost more and show a higher implied volatility.
- Remember one rule: a heavy tail raises the implied volatility of strikes in that tail.
- It follows that pricing every strike with the at the money volatility misprices the tails.

11 A heavy left tail behind the skew



IN THE LESSON

- Here is the distribution behind an equity skew.
- Across the bottom is the index price in six months, around today's one hundred, and up the side is probability density.
- The grey blue curve is a lognormal distribution using the at the money volatility, the flat smile.
- The red curve is an illustration of what the market implies.
- Look at the left.
- The red curve lies above the lognormal at low prices, so the market gives more probability to a crash.
- That is the heavier left tail.
- On the right, the red curve lies below, a thinner right tail.
- Backing implied volatilities out of the red curve gives a downward skew.

12 Tail shape, smile shape, dearer options

Compared with a lognormal at the at-the-money volatility

Distribution	Smile shape	Dearer options
Heavy left, light right tail	Downward skew	OTM puts
Both tails heavy, higher peak	U-shaped smile	OTM puts and calls
Heavy right tail	Upward skew	OTM calls
Lognormal	Flat	None

IN THE LESSON

- Here is the whole rule as a table, read row by row.
- In the first row, a heavier left tail and a lighter right tail than the lognormal give a downward skew, so out of the money puts are dearer than the model at the at the money volatility, and out of the money calls are cheaper.
- In the second row, heavier tails on both sides, with a higher peak, give a U shaped smile, and both out of the money calls and puts are dearer.
- In the third row, a heavier right tail gives an upward sloping skew, so out of the money calls are dearer.
- In the last row, the lognormal itself gives a flat line, and the model prices every strike correctly.

13 Example 3: pricing the tails with one volatility

Index 100, 6 months, r = 4%. Skew: 28% at strike 80, 17% at strike 115

80 put: BSM at 20% vs 28%		**$0.22 vs $0.93**
80 put mispricing	0.93 − 0.22	**$0.71 too cheap**
115 call: BSM at 20% vs 17%		**$1.66 vs $1.07**
115 call mispricing	1.66 − 1.07	**$0.59 too dear**

IN THE LESSON

- Back to the analyst's problem, now in numbers.
- The index is at one hundred, six months, rate four per cent.
- Kestrel's skew gives twenty eight per cent at a strike of eighty and seventeen per cent at one hundred and fifteen.
- The eighty strike put priced at the at the money volatility of twenty per cent is worth twenty two cents.
- At its own implied volatility of twenty eight per cent it is worth ninety three cents.
- So the single volatility underprices it by seventy one cents.
- The one hundred and fifteen strike call at twenty per cent is worth one dollar sixty six, but at seventeen per cent only one dollar seven.
- So it is overpriced by fifty nine cents.

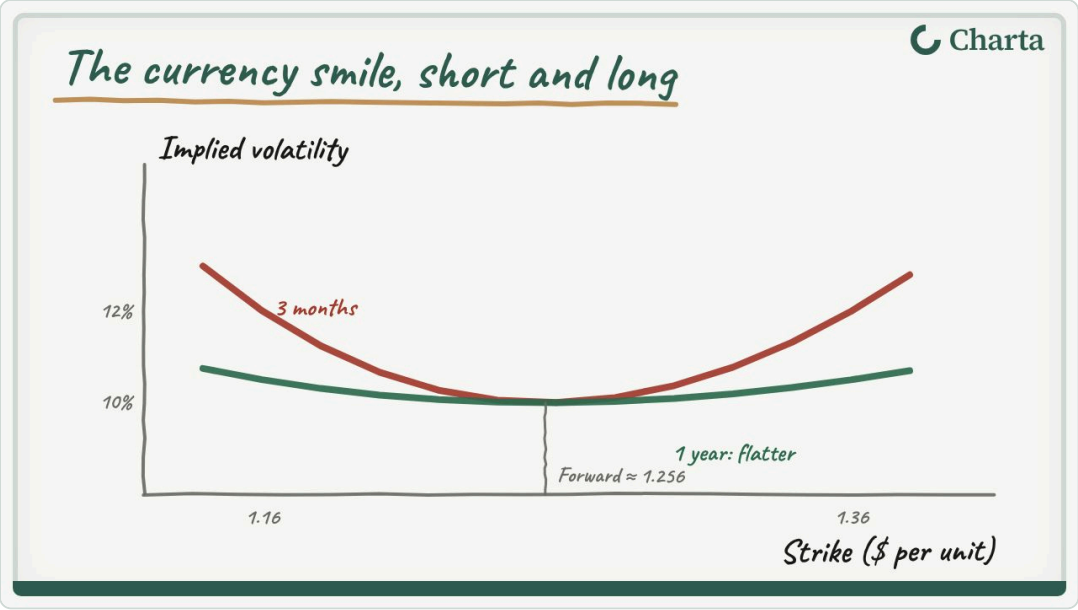
14 How currency returns are distributed

- Volatility changes randomly over time
- Rates sometimes jump: intervention, political shocks
- Fatter tails, higher peak: positive excess kurtosis
- Little asymmetry: one currency's fall is the other's rise
- Result: a symmetric U, lowest near the forward

IN THE LESSON

- Now to currencies.
- Exchange rate returns are not lognormal with constant volatility, for two reasons.
- First, volatility changes randomly over time.
- Second, exchange rates sometimes jump, for example when a central bank intervenes or a political shock occurs.
- Both features put more probability than a lognormal in both tails, and they give a higher peak in the middle.
- A distribution like that has positive excess kurtosis, meaning fatter tails than a normal distribution.
- There is little asymmetry for a major currency pair, because a fall in one currency is a rise in the other.
- The result is a symmetric U shaped smile, lowest near the at the money, or forward, strike.

15 The currency smile, short and long



IN THE LESSON

- Here is the currency smile on Kestrel's foreign exchange desk, for a major pair quoted at one point two five dollars per unit.
- Across the bottom is the strike, and up the side is implied volatility.
- The red curve is the three month smile.
- Its lowest point is ten per cent, near the forward rate of about one point two five six, and it rises to twelve per cent at strikes of one point one six and one point three six.
- The green curve is the one year smile, which is much flatter.
- The labels read: three months, one year, flatter, and the forward.
- The effect of fat tails is strongest for short maturities and fades as maturity lengthens.

16 Example 4: what one volatility does to FX options

Spot 1.25, 3 months, $ rate 4%, foreign rate 2%. Per 100,000 units

Put 1.16: BSM at 10% vs 12%	141 vs 307	**short by $166**
Call 1.36: BSM at 10% vs 12%	155 vs 337	**short by $182**
Strangle: put + call	296 vs 644	**short by $348**
Risk reversal: call − put	14 vs 30	**off by $16**

IN THE LESSON

- Price two three month options on one hundred thousand units of the currency, with a dollar rate of four per cent and a foreign rate of two per cent.
- The put with a strike of one point one six, valued at the at the money volatility of ten per cent, is worth one hundred and forty one dollars, but at its smile volatility of twelve per cent, three hundred and seven.
- Short by one hundred and sixty six.
- The call at one point three six is worth one hundred and fifty five against three hundred and thirty seven, short by one hundred and eighty two.
- A strangle, long both, is short by three hundred and forty eight dollars.
- A risk reversal, long the call and short the put, is worth fourteen against thirty, off by only sixteen, because the two wing errors largely cancel.
- So one volatility badly undervalues a strangle, while the error on a risk reversal is small.

17 Equity against currency: the patterns

Match each asset class to its pattern, and know the reasons

	Equity and index	Currencies
Pattern	Downward skew	Symmetric U smile
Tails vs lognormal	Heavy left, thin right	Both heavy, higher peak
Highest implied vol	Low-strike OTM puts	Both wings
Why	Leverage, crashophobia	Random vol, jumps

IN THE LESSON

- Here are the two asset classes side by side, row by row.
- The pattern for equity and equity index options is a downward skew, while currency options show a roughly symmetric U shaped smile.
- Compared with a lognormal, equities have a heavy left tail and a thin right tail, while currencies have both tails heavy and a higher peak.
- So the highest implied volatility for equities is at low strikes, on out of the money puts, while for currencies it is on both wings.
- The last row gives the reasons.
- For equities there are two explanations, the leverage effect and crashophobia.
- For currencies, volatility moves randomly and rates jump, with no natural crash side.

18 Example 5: the leverage effect

Assets $100m, debt $40m, equity $60m, asset vol 15%. Simplification: $\sigma E \times E = \sigma A \times V$

Equity vol today	15% × 100 ÷ 60	**25%**
Assets fall 20%	100 → 80; debt stays 40	**equity $40m**
Equity vol after	15% × 80 ÷ 40	**30%**
Share price change	(40 − 60) ÷ 60	**−33.3%**

IN THE LESSON

- The first equity explanation is the leverage effect.
- When a share price falls, the firm's debt to equity ratio rises, the equity becomes riskier, and its volatility increases.
- So volatility moves against the price.
- A simplification used in questions: equity volatility times equity value equals asset volatility times asset value.
- A firm has assets of one hundred million dollars, debt of forty and equity of sixty, and its assets have a volatility of fifteen per cent.
- Equity volatility is fifteen per cent times one hundred over sixty, twenty five per cent.
- Now the assets fall by twenty per cent to eighty, while debt stays at forty, so equity falls to forty.
- Equity volatility becomes fifteen per cent times eighty over forty, thirty per cent.
- The share price fell by a third and its volatility rose.

19 Crashophobia, and why currencies have no skew

- Crashophobia: since Oct 1987, investors pay for OTM puts
- The skew was not pronounced before that date
- Both stories: heavy left tail, vol up as prices fall
- Currencies: no natural crash direction
- So no skew: the smile stays roughly symmetric

IN THE LESSON

- The second explanation is crashophobia, a fear of crashes.
- Since the stock market crash of October nineteen eighty seven, investors have been willing to pay up for out of the money puts to guard against another crash.
- That demand lifts their implied volatilities.
- The equity skew was not pronounced before that date, which supports the explanation.
- Both stories give a heavy left tail, with volatility rising as prices fall.
- Currencies are different.
- A major exchange rate has no natural crash direction, because a fall in one currency is a rise in the other.
- So no skew arises, and the smile stays roughly symmetric.

20 Other ways to describe the smile

- Strike K, or moneyness K/S0
- Forward moneyness K/F0, with F0 = S0 × e^{(r − q)T}
- Or the option's delta
- Against fixed strikes, the smile shifts when spot moves
- Against moneyness or delta it is far more stable

IN THE LESSON

- So far we have plotted implied volatility against the strike, K.
- Traders often use other variables.
- One is moneyness, the strike divided by today's price, K over S nought.
- Another is forward moneyness, K over F nought, where F nought is the forward price.
- With r the domestic rate and q the dividend yield or foreign rate, the forward is S nought times e to the r minus q, times T.
- A third choice is the option's delta.
- Why bother?
- Stability.
- When spot moves, a smile drawn against fixed strikes shifts sideways.
- The same smile drawn against moneyness or delta changes much less over time, and it is easier to compare across assets and across dates.

21 Example 6: moneyness and forward moneyness

Index 100, 1 year, strike 95, r = 4%, dividend yield q = 1%

Forward price F0	100 × e^{(0.04 − 0.01) × 1}	**103.05**
Moneyness K/S0	95 ÷ 100	**0.95**
Forward moneyness K/F0	95 ÷ 103.05	**0.922**
Spot to 110: same 0.95	0.95 × 110	**K = 104.5**

IN THE LESSON

- Take Kestrel's index at one hundred, a one year option struck at ninety five, a rate of four per cent and a dividend yield of one per cent.
- The forward price is one hundred times e to the power of four per cent minus one per cent, times one year, which is one hundred and three dollars and five cents.
- Moneyness is ninety five over one hundred, point nine five.
- Forward moneyness is ninety five over one hundred and three point zero five, about point nine two two.
- Now the index jumps to one hundred and ten.
- If the smile is stable in moneyness, the volatility of point nine five now belongs to a strike of point nine five times one hundred and ten, one hundred and four point five.
- The smile moved with the market.

22 How FX desks quote the smile

RR = σ(25Δ call) − σ(25Δ put)
Strangle = [σ(25Δ call) + σ(25Δ put)] ÷ 2 − σ(ATM)

At-the-money volatility	Level of the smile
Risk reversal	Skew: calls minus puts
Strangle	Curvature: wings above middle

IN THE LESSON

- Currency desks quote the smile with three numbers, using delta rather than strike.
- The first is the at the money volatility, the level of the smile.
- The second is the risk reversal: the volatility of a call minus the volatility of a put with the same delta, usually twenty five delta.
- It measures skew.
- The third is the strangle: the average of the call and put volatilities, minus the at the money volatility.
- It measures curvature, how much the wings sit above the middle.
- Remember parity here.
- A twenty five delta put and a seventy five delta call at the same strike share one implied volatility, so they use the same number.

23 Example 7: from quotes to volatilities

Quotes: ATM 9.0%, 25-delta risk reversal +0.6%, strangle 0.4%

Average of the wings	ATM + strangle = 9.0 + 0.4	**9.4%**
Half the risk reversal	0.6 ÷ 2	**0.3%**
25-delta call vol	9.4 + 0.3	**9.7%**
25-delta put vol	9.4 − 0.3	**9.1%**
Check	9.7 − 9.1 = 0.6; average 9.4	**as quoted**

The 75-delta call at the 25-delta put's strike also uses 9.1%

IN THE LESSON

- Kestrel's currency desk sees these quotes on another pair: at the money nine per cent, a twenty five delta risk reversal of plus point six, and a strangle of point four.
- What are the wing volatilities?
- The strangle says the average of the two wings is nine plus point four, nine point four per cent.
- The risk reversal says the call sits above the put by point six, so each wing is half of that, point three, away from the average.
- The twenty five delta call volatility is nine point four plus point three, nine point seven per cent.
- The twenty five delta put volatility is nine point four minus point three, nine point one per cent.
- Check: the difference is point six and the average is nine point four, as quoted.

24 The volatility term structure

- Term structure: ATM implied vol against maturity
- Volatility mean-reverts to a long-run level
- Short-term vol low: the curve slopes up
- Short-term vol high, after turmoil: it slopes down
- Surface = smile + term structure: a vol for each K and T

IN THE LESSON

- Now we add maturity.
- The volatility term structure is the implied volatility of at the money options plotted against maturity.
- Its shape follows from one fact: volatility mean reverts, meaning it tends to drift back to a long run level.
- When short term volatility is low relative to that level, the curve slopes upward, because the market expects volatility to rise.
- When short term volatility is unusually high, for example after market turmoil, the curve slopes downward, because the market expects it to calm down.
- Put the smile and the term structure together and you get the volatility surface.
- It gives an implied volatility for each strike, moneyness or delta, and each maturity.

25 Kestrel's volatility surface

Implied volatility, %, by maturity and strike. Index at 100

Maturity	K = 90	K = 100	K = 110
3 months	25.0	19.5	17.0
6 months	23.5	20.0	18.0
1 year	22.0	20.5	19.3
2 years	21.2	20.8	20.3

IN THE LESSON

- Here is Kestrel's index volatility surface, in per cent, for strikes of ninety, one hundred and one hundred and ten.
- At three months: twenty five, nineteen point five and seventeen.
- At six months: twenty three point five, twenty and eighteen.
- At one year: twenty two, twenty point five and nineteen point three.
- At two years: twenty one point two, twenty point eight and twenty point three.
- Down the middle column, the at the money volatility rises from nineteen point five to twenty point eight, an upward term structure.
- Across the rows, the gap between strikes ninety and one hundred and ten shrinks from eight points to under one.
- The smile is most pronounced at short maturities and flatter at long ones.

26 Example 8: pricing off the surface

Wanted: 9-month put, strike 95. Interpolate linearly in strike, then in maturity

6 months, K = 95	(23.5 + 20.0) ÷ 2	**21.75%**
1 year, K = 95	(22.0 + 20.5) ÷ 2	**21.25%**
9 months, K = 95	(21.75 + 21.25) ÷ 2	**21.50%**
Put price via BSM	S0 100, r 4%, σ 21.5%	**$3.89**

Interpolate volatilities, never prices

IN THE LESSON

- A client asks Kestrel for a nine month put with a strike of ninety five.
- Neither is on the surface, so we interpolate linearly in strike and in maturity.
- At six months, ninety five is halfway between ninety and one hundred, so the volatility is the average of twenty three point five and twenty, which is twenty one point seven five per cent.
- At one year, the average of twenty two and twenty point five is twenty one point two five.
- Nine months is halfway between six months and a year, so the volatility is twenty one point five per cent.
- Put into the B S M formula, with the index at one hundred and a rate of four per cent, the put is worth three dollars eighty nine.
- Interpolate volatilities, never prices.

27 Example 9: forward volatility from the term structure

Forward variance = (σ2^2 × T2 − σ1^2 × T1) ÷ (T2 − T1). Kestrel's ATM vols

6 months: σ1^2 × T1	0.20^2 × 0.5	**0.0200**
1 year: σ2^2 × T2	0.205^2 × 1 = 0.042025	**0.0420**
Forward variance	(0.042025 − 0.02) ÷ 0.5	**0.04405**
Forward volatility	√0.04405	**20.99%**
Wrong: subtract the vols	20.5 − 20.0	**0.5%**

IN THE LESSON

- What volatility does the market expect between six months and one year?
- Variances add over time, volatilities do not.
- The six month at the money volatility is twenty per cent, so its variance times time is point two squared times point five, point zero two.
- The one year volatility is twenty point five per cent, so point two zero five squared times one, point zero four two zero two five.
- The forward variance is the difference over the half year between them: point zero two two zero two five over point five, which is point zero four four zero five.
- Its square root is a forward volatility of about twenty point nine nine per cent.
- The wrong method subtracts the volatilities, twenty point five minus twenty, and gets half a per cent, which is meaningless.

28 Delta when the smile moves

Δ(effective) = Δ(BSM) + Vega × ∂σ/∂S
Sticky strike: ∂σ/∂S = 0

Effective delta	What the hedge really needs
BSM delta	Model delta at the option's vol
Vega	Value change per vol point
Change in implied vol per $1	How the smile moves with S

IN THE LESSON

- The Greeks of the B S M model assume constant volatility.
- When implied volatility varies with strike and maturity, and moves when the asset price moves, delta needs an extra term.
- The effective delta is the model delta plus vega times the change in implied volatility per unit change in the asset price.
- Vega is the change in option value for a change in volatility.
- If each option's implied volatility stays fixed whatever the price, which is called sticky strike, the change is zero and the extra term vanishes.
- If instead the smile is a function of K over S, the volatility of a given option changes as S moves, and the extra term is not zero.

29 Example 10: three deltas for one call

6-month ATM call, S0 = K = 100, σ 20%. Skew: 23.5% at 90 to 18.0% at 110

BSM delta and vega	vega per vol point 0.276	**Δ 0.584**
Sticky strike	0.584 + 0.276 × 0	**0.584**
Skew slope per $1 of strike	(18.0 − 23.5) ÷ 20	**−0.275 pt**
Smile in K/S: σ up 0.275	0.584 + 0.276 × 0.275	**0.660**
Vol falls 0.1 pt per $1 rise	0.584 − 0.276 × 0.1	**0.556**

IN THE LESSON

- Kestrel holds the six month call with index and strike at one hundred and volatility at twenty per cent.
- Its model delta is point five eight four and its vega is point two seven six dollars per volatility point.
- Under sticky strike the change in volatility is zero, so delta stays point five eight four.
- Now suppose the smile is a function of K over S.
- Kestrel's six month skew falls from twenty three point five at ninety to eighteen at one hundred and ten, minus point two seven five points per unit of strike.
- A one dollar rise in S lowers K over S, like a one point lower strike, so this option's volatility rises point two seven five points, pushing the call delta up to point six six zero.
- If instead observed volatility falls by a tenth of a point for each dollar rise, the effective delta is lower, point five five six.

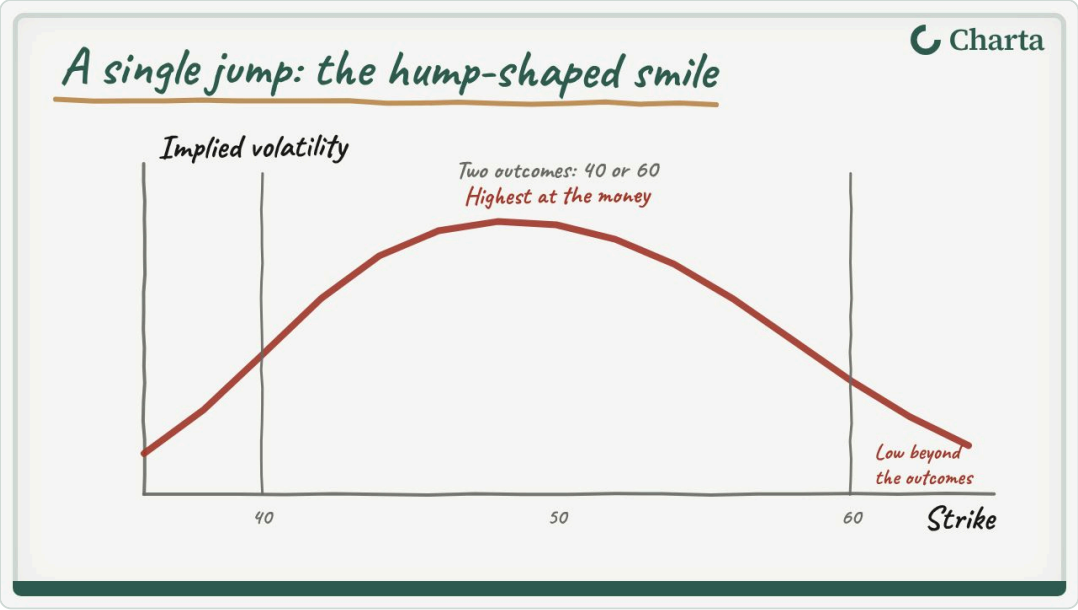
30 Delta and vega on a moving smile

- Skew in K/S: price up, vol up, call delta up
- Observed vol falls as price rises: delta below model
- Always state how the smile is assumed to move
- One parallel vega hides strike and maturity exposures
- Measure vega bucket by bucket across the surface

IN THE LESSON

- Two lessons follow.
- First, keep the two cases apart.
- With a downward skew in K over S, a rise in the price raises the option's volatility and pushes the call delta up.
- When observed volatility instead falls as the price rises, the effective delta is below the model delta.
- So always state which assumption about the smile's movement you are using.
- Second, vega.
- A single vega measures a parallel shift of the whole surface.
- That hides exposure to moves at particular strikes and maturities, for example a steepening of the short dated skew while long dated volatility stays put.
- So risk is measured bucket by bucket, with a vega for each region of strike and maturity.

31 A single jump: the hump-shaped smile



IN THE LESSON

- Now a different shape.
- Kestrel trades options on a single stock at fifty, and a court ruling is due.
- The market expects the ruling to move the stock to either forty or sixty.
- The price distribution at expiry is then bimodal, with two peaks, little probability in the middle, and thinner tails than a lognormal.
- Strikes beyond the two outcomes are worth very little, so their implied volatility is low.
- Options near the money carry all the jump risk.
- The curve shows implied volatilities from pricing options on those two outcomes, with a little ordinary volatility around each.
- The red curve is hump shaped.
- The labels read: two outcomes, forty or sixty, then highest at the money, then low beyond the outcomes.
- That is the reverse of the U shaped smile of fat tails.

32 Example 11: a jump adds variance

Ordinary vol 20% a year. Jump ≈ ±20%, variance ≈ 0.2^2 = 0.04. Ruling in 2 months

1 month: expires before	no jump	**20.0%**
3 months	√((0.04 × 0.25 + 0.04) ÷ 0.25)	**44.7%**
1 year	√((0.04 × 1 + 0.04) ÷ 1)	**28.3%**

Biggest at short maturities; the term structure steps up after the event

IN THE LESSON

- A jump adds its variance to the total variance over an option's life, so variances add, not volatilities.
- Take ordinary volatility of twenty per cent a year, and a jump of about twenty per cent either way, so its variance is roughly point two squared, point zero four.
- The ruling comes in two months.
- A one month option expires before the event and keeps an implied volatility of twenty per cent.
- A three month option has ordinary variance of point zero four times a quarter, point zero one, plus the jump's point zero four, so point zero five in all.
- Divided by a quarter and square rooted, about forty four point seven per cent.
- A one year option has point zero eight in all, about twenty eight point three per cent.
- So the jump matters most for short maturities, and the term structure steps up after the event.

33 Mistakes to avoid

- Smile: across strikes. Term structure: across maturities
- Same strike and expiry: one implied vol for call and put
- Heavy tail raises the vol there: equity skew, FX U
- Interpolate vols, not prices; add variances, not vols
- Delta: state the smile's move. One jump: a hump, not a U

IN THE LESSON

- Here are the mistakes that cost marks.
- First, the smile varies the strike and the term structure varies the maturity, so do not mix them up.
- Second, a European call and put with the same strike and expiry have one implied volatility; a difference means arbitrage.
- Third, a heavy tail raises the implied volatility of strikes in that tail, so equities show a downward skew and currencies a U.
- Fourth, interpolate volatilities, not prices, and add variances, not volatilities, when you work out a forward volatility.
- Fifth, when you adjust delta for the smile, state how the smile is assumed to move.
- And remember that a single expected jump gives a hump, highest at the money, not a U.

34 Try three quick questions

1. ATM vol 18% at 3 months, 21% at 1 year. Forward vol between them?
   Answer: (0.0441 − 0.0081) ÷ 0.75 = 0.048; √ ≈ 21.9%.
2. 6-month European put: implied vol 31%. The call, same strike and expiry?
   Answer: 31%, by put-call parity.
3. ATM 8.0%, 25-delta RR −0.4%, strangle 0.3%. 25-delta put vol?
   Answer: 8.0 + 0.3 − (−0.4 ÷ 2) = 8.5%.

IN THE LESSON

- Now try three quick questions.
- Pause, think, then listen.
- Question one.
- The three month at the money volatility is eighteen per cent and the one year is twenty one per cent.
- What is the forward volatility between them?
- Variance times time is point zero four four one and point zero zero eight one.
- The difference, point zero three six, over three quarters of a year is point zero four eight, so the forward volatility is about twenty one point nine per cent.
- Question two.
- A six month European put on a share has an implied volatility of thirty one per cent.
- What is the call's, with the same strike and expiry?
- Thirty one per cent, by put call parity.
- Question three.
- A currency quotes at the money eight per cent, a twenty five delta risk reversal of minus point four and a strangle of point three.
- What is the twenty five delta put volatility?
- Eight plus point three, minus half of minus point four, eight point five per cent.