CHARTA LEARNING · LESSON NOTES

Percents in business, mixtures and weighted averages

GMAT Quantitative Reasoning

1 A story: up twenty, down twenty

- +20% of $100 → $120
- −20% of $120 → $96, not $100
- Each percent has its own base
- First habit: percent of what?

IN THE LESSON

- A shop owner raises a price by twenty per cent.
- Business is slow, so she cuts the new price by twenty per cent.
- Is the price back where it began?
- Many people say yes.
- Try it with a hundred dollars.
- Up twenty per cent is a hundred and twenty.
- Twenty per cent of a hundred and twenty is twenty four, so the price falls to ninety six.
- The price ended four dollars lower.
- The two changes used different bases.
- The rise was measured on the old price, and the fall on the new one.
- Nearly every wrong answer in money questions comes from this one slip, so our first habit is always to ask: twenty per cent of what?

2 Cost, revenue, profit and the base

- Profit = revenue − cost
- Profit % and markup: base is cost
- Discount: base is the marked price
- Margin: base is revenue

IN THE LESSON

- Let's define the words.
- Cost is what the seller paid.
- Revenue is what the seller receives from customers, which is the selling price times the number sold.
- Profit is revenue minus cost.
- If it is negative, it is a loss.
- Now the base.
- A profit percent or a markup is usually measured on cost, unless the question says otherwise.
- A discount is measured on the marked price, the price on the tag.
- And a margin is measured on revenue.
- Same dollars, different percents, depending on what you divide by.
- A good exam habit is to underline the base word in the question before you do anything else.

3 The profit formulas

Profit = Revenue − Cost
Profit % of cost = Profit ÷ Cost × 100

Revenue (R)	Money coming in
Cost (C)	Money paid out
Profit (P)	Revenue minus cost
Profit percent on cost	Profit ÷ cost, times 100

IN THE LESSON

- Here are the two formulas.
- Profit equals revenue minus cost.
- Take the terms one at a time.
- Revenue is the money coming in.
- Cost is the money paid out.
- Profit is the difference.
- The second formula says profit as a percent of cost equals the profit divided by the cost, times one hundred.
- Notice that the cost is on the bottom.
- If a question asks for profit as a percent of the selling price, then revenue goes on the bottom instead.
- The numbers on top stay the same.
- Only the bottom changes, and that single swap produces most of the wrong answer choices.

4 Cost, marked price, selling price

Percent of what? Follow the arrows

Cost: 80 → Marked: 100 (25% of 80 added) → Sold: 90 (10% of 100 off)

IN THE LESSON

- Here is a picture to keep in your head.
- Three boxes, left to right.
- On the left, the cost: say eighty dollars.
- A shop adds twenty five per cent of the cost, which is twenty dollars, and writes one hundred dollars on the tag.
- That is the marked price.
- Then the shop gives ten per cent off the marked price, which is ten dollars, and sells for ninety.
- Look at what each percent is measured on.
- The markup of twenty five per cent is measured on the eighty.
- The discount of ten per cent is measured on the hundred.
- The profit is ten dollars, which is twelve and a half per cent of the cost.

5 Example 1: a plain profit question

Cost $60, selling price $75. Profit as a percent of cost?

Profit	75 − 60	**$15**
Divide by the cost	15 ÷ 60 = 1/4	**25%**
Trap: divide by price	15 ÷ 75	**20%**

Same $15, different base, different percent

IN THE LESSON

- Example one, the plain type.
- A retailer buys a lamp for sixty dollars and sells it for seventy five.
- The profit is what percent of the cost?
- Step one, profit is seventy five minus sixty, which is fifteen.
- Step two, the question says percent of cost, so cost goes on the bottom.
- Fifteen over sixty is one quarter, which is twenty five per cent.
- Now the trap.
- If you divided by the selling price, you would get fifteen over seventy five, which is twenty per cent, and twenty per cent will be sitting there among the choices.
- Always read the base.

6 Working backwards with multipliers

- Markup 50% → multiply by 1.5
- Discount 20% → multiply by 0.8
- Chain: multiply the multipliers
- Find the start: divide by the chain

IN THE LESSON

- When a question gives you the final price and asks for the cost, work backwards with multipliers.
- A multiplier is just the percent turned into a decimal factor.
- A fifty per cent markup means multiply by one point five.
- A twenty per cent discount means multiply by zero point eight, because you keep eighty per cent.
- Chain the multipliers together by multiplying them.
- To recover the starting number, divide the final price by the combined multiplier.
- Never subtract percents from each other.
- A markup and a discount are measured on different bases, so they cannot be netted.
- They multiply.
- This is the fastest sound method, and it makes the arithmetic short.

7 Example 2: a markup and a discount

Markup 50% on cost, then 20% off, sold for $168. Cost?

Multipliers	1.5 and 0.8	
Combined	1.5 × 0.8	**1.2**
Cost	168 ÷ 1.2	**$140**
Check	140 → 210 → 168	**works**
Trap: net 50 − 20 = 30	168 ÷ 1.3	**$129.23**

IN THE LESSON

- Example two, the typical type.
- A shop marks a jacket up fifty per cent on its cost, then sells it at twenty per cent off the marked price for one hundred and sixty eight dollars.
- What did the jacket cost the shop?
- Step one, the multipliers are one point five and zero point eight.
- Step two, multiply them: one point five times zero point eight is one point two.
- Step three, the selling price is one point two times the cost, so the cost is one hundred and sixty eight divided by one point two, which is one hundred and forty dollars.
- Check it: one hundred forty times one point five is two hundred ten, and twenty per cent off two hundred ten leaves one hundred sixty eight.
- The trap is netting fifty against twenty to get thirty.

8 Revenue, costs and a profit target

- Variable cost: grows with each unit
- Fixed cost: paid whatever you sell
- Profit = units × (price − variable) − fixed
- Break even: profit equals zero

IN THE LESSON

- Some questions build profit from parts.
- Think of a stall that sells lemonade.
- Each cup brings in a price and costs a little to make.
- That per cup cost is called the variable cost, because it grows with the number of cups.
- Then there is the stall rental, which you pay however many cups you sell.
- That is the fixed cost.
- The money left from each cup after its own cost is the profit per cup.
- Multiply by the number of cups to get total contribution, then subtract the fixed cost for the actual profit.
- If a target profit is given, set the profit equal to the target and solve for the number of units.
- Break even means the target is zero.

9 The profit-per-unit formula

Profit = U × (Price − Variable cost) − Fixed cost

Units sold (U)	How many units sold
Price per unit	Selling price of one unit
Variable cost per unit	Cost of making one unit
Fixed cost	Paid whatever you sell

IN THE LESSON

- Here is the formula for the build-up.
- Profit equals the number of units, times the price minus the variable cost, then minus the fixed cost.
- Terms first.
- U is the units sold.
- The price is what each unit sells for.
- The variable cost is what each unit costs to make.
- The fixed cost is the lump sum you pay regardless.
- The bracket, price minus variable cost, is the amount each unit adds.
- To hit a target profit, add the fixed cost to the target, then divide by the amount each unit adds.
- That is the number of units you must sell.

10 Example 3: how many units for a target?

Price $25, variable cost $15, fixed cost $4,000, target profit $2,000

Added by each mug	25 − 15	**$10**
Must be covered	2,000 + 4,000	**$6,000**
Mugs needed	6,000 ÷ 10	**600**
Break even, for contrast	4,000 ÷ 10	**400 mugs**

IN THE LESSON

- Example three.
- A workshop sells mugs for twenty five dollars each.
- Each mug costs fifteen dollars to make, and the workshop pays four thousand dollars fixed costs.
- How many mugs must it sell to make a profit of two thousand dollars?
- Step one, each mug adds twenty five minus fifteen, which is ten dollars.
- Step two, the target plus the fixed cost is two thousand plus four thousand, six thousand dollars that the mugs must cover.
- Step three, six thousand divided by ten is six hundred mugs.
- Check: six hundred mugs bring in fifteen thousand, costs are nine thousand variable plus four thousand fixed, thirteen thousand, and the profit is two thousand.
- The trap is forgetting the fixed cost and answering two hundred.

11 Profit rises with every mug

Chart: Profit ($) vs Mugs sold (up to 800)

- Break even: 400
- Target: 600 mugs
- Start: −4,000
- $0

IN THE LESSON

- Here is Example three as a picture.
- Along the bottom, the number of mugs sold.
- Up the side, the profit in dollars.
- The line starts at the bottom left, where nothing has been sold and the workshop is four thousand dollars down, because the fixed cost has been paid.
- The line climbs ten dollars for every mug.
- The grey horizontal line in the middle marks zero profit.
- The line crosses it at four hundred mugs, which is break even.
- The upper grey line marks the target of two thousand dollars.
- The line reaches it at six hundred mugs.
- The picture shows why the answer is a straight division.

12 Simple interest: the same amount each year

- Principal: the money you start with
- Simple interest: always on the original
- Interest = principal × rate × years
- Balance = principal + interest

IN THE LESSON

- Now interest.
- Interest is the price of borrowing money, or the reward for lending it.
- The money you start with is called the principal.
- The rate is a percent per year.
- With simple interest, the interest is always worked out on the original principal and nothing else.
- So a deposit earns the same dollars each year, like a tap with a steady drip.
- Multiply principal by rate by the number of years, and that is the interest.
- If the question asks for the final balance, add the principal back.
- The exam often asks for interest only, and the wrong answer choices include the balance, so check which one is wanted before you finish.

13 Simple and compound interest

Simple: I = P × r × t
Compound: A = P × (1 + r/n)^(n×t)

Principal (P)	Starting money
Rate (r)	Yearly rate as a decimal
Years (t)	Number of years
Periods a year (n)	Times a year interest is added
Amount (A)	Balance after t years

IN THE LESSON

- Two formulas.
- Simple interest equals principal times rate times years.
- The principal is the starting money.
- The rate is written as a decimal, so six per cent is zero point zero six.
- And t is the years.
- The second formula is for compound interest, where the interest also earns interest.
- The amount after t years equals the principal times one plus r over n, all raised to the power n times t.
- Here n is how many times a year the interest is added.
- For a twice a year bank, n is two.
- Each period multiplies the balance by one plus r over n, once for every period.
- Count the periods, multiply that many times.

14 Example 4: simple against compound

$5,000 at 6% a year for 3 years

Simple interest	5,000 × 0.06 × 3	**$900**
Compound factor	1.06 × 1.06 × 1.06	**1.191016**
Compound balance	5,000 × 1.191016	**$5,955.08**
Compound interest	5,955.08 − 5,000	**$955.08**
Interest on interest	955.08 − 900	**$55.08**

IN THE LESSON

- Example four.
- An investor puts in five thousand dollars at six per cent a year for three years.
- First, simple interest.
- Five thousand times zero point zero six is three hundred a year.
- Three years gives nine hundred dollars.
- Now compound once a year.
- Each year multiplies the balance by one point zero six.
- After three years that is one point zero six cubed, which is one point one nine one zero one six.
- Multiply by five thousand and the balance is five thousand nine hundred fifty five dollars and eight cents.
- The interest is nine hundred fifty five dollars and eight cents.
- Compound beats simple by fifty five dollars and eight cents, the interest earned on interest.
- On the exam, you would estimate: three times three hundred, plus a little extra.

15 Interest on interest, drawn

Interest earned ($)

Year one	**400**
Year two	**440**
Year three	**484**
Simple, any year	**400**

$4,000 at 10% a year, compounded yearly

IN THE LESSON

- Look at how compound interest grows.
- Take four thousand dollars at ten per cent a year, compounded once a year.
- The bars show the interest earned in each year.
- In year one, ten per cent of four thousand is four hundred dollars.
- At the end, the balance is four thousand four hundred.
- In year two, ten per cent of four thousand four hundred is four hundred and forty.
- In year three, ten per cent of four thousand eight hundred forty is four hundred and eighty four.
- Each bar is taller than the last, because the base keeps growing.
- With simple interest, every bar would stay at four hundred.
- That growing base is the whole difference.

16 Example 5: the half-year trap

$2,000 at 12% a year, compounded twice a year, 1 year

Rate per period	12% ÷ 2	**6%**
Factor for the year	1.06 × 1.06	**1.1236**
Balance	2,000 × 1.1236	**$2,247.20**
Interest	2,247.20 − 2,000	**$247.20**
Trap: 12% each half year	2,000 × 1.12^2 − 2,000	**$508.80**

IN THE LESSON

- Example five, the trap type.
- A deposit of two thousand dollars earns twelve per cent a year, compounded twice a year.
- How much interest in one year?
- The rate per period is twelve divided by two, six per cent, and there are two periods.
- Step one, the factor is one point zero six, twice.
- One point zero six squared is one point one two three six.
- Step two, the balance is two thousand times one point one two three six, which is two thousand two hundred forty seven dollars and twenty cents.
- Step three, the interest is two hundred forty seven dollars and twenty cents.
- Two wrong choices wait for you.
- Simple interest gives two hundred forty.
- And using the full twelve per cent each half year gives five hundred eight dollars.

17 Mistakes to avoid in money questions

- 1. Profit and markup: base is cost
- 2. Markup and discount multiply, never net
- 3. Interest or balance? Read what is asked
- 4. Compound: rate ÷ periods, not full rate
- 5. Profit target: include fixed costs

IN THE LESSON

- Time for the mistakes that cost marks.
- One.
- Using the selling price as the base for a profit or markup.
- Unless told otherwise, markup and profit are on cost.
- Two.
- Netting a markup against a discount by subtraction.
- They have different bases, so multiply the multipliers.
- Three.
- Answering with the balance when the question asks for interest, or the other way round.
- Four.
- Using the full yearly rate for each half year in compound interest.
- Divide the rate by the number of periods.
- Five.
- Forgetting fixed costs when a profit target is given.
- Take a breath, underline the base, and the traps lose their power.

18 Part two: mixtures and weighted averages

- Averaging averages ignores group sizes
- Bigger group, bigger pull
- Result sits nearer the larger group
- Total of the ingredient ÷ total size

IN THE LESSON

- Part two.
- Imagine a class of thirty students with an average score of sixty, joining a class of ten with an average score of ninety.
- What is the average of all forty?
- If you said seventy five, you averaged the averages, and you ignored the sizes.
- The big class has three quarters of the students, so it pulls much harder.
- The true answer is closer to sixty.
- That pull is the weight.
- A weighted average counts each group by its size, so the result always lands nearer the bigger group.
- The same idea runs through every mixture, whether it is juice, salt water, or prices of tickets.
- Find the total of the ingredient, and divide by the total size.

19 The weighted average formula

Weighted average = (A size × A average + B size × B average) ÷ (A size + B size)

Size of group A	How many in group A
Average of group A	Average of group A
Size of group B	How many in group B
Average of group B	Average of group B
Weighted average	Total ÷ total size

IN THE LESSON

- Here is the formula.
- The weighted average equals the size of group A times its average, plus the size of group B times its average, all divided by the two sizes added together.
- Take the terms.
- The size of group A is how many items, or how many litres, it holds.
- Its average is the percent or score per item.
- Multiplying the two gives the total amount of the ingredient in that group.
- Do the same for group B, add the totals, and divide by the combined size.
- The top is the total ingredient, the bottom is the total size.
- For a mixture, the average is the strength, such as the percent of salt.

20 Example 6: mixing two syrups

8 L of 15% syrup + 12 L of 40% syrup. Strength?

Sugar in the first	0.15 × 8	**1.2 L**
Sugar in the second	0.40 × 12	**4.8 L**
Total sugar, total mix	6 L in 20 L	
Strength	6 ÷ 20	**30%**
Trap: plain average	(15 + 40) ÷ 2	**27.5%**

IN THE LESSON

- Example six, the plain type.
- Eight litres of a fifteen per cent sugar syrup are mixed with twelve litres of a forty per cent syrup.
- What is the strength of the mixture?
- Step one, the sugar in the first syrup is fifteen per cent of eight, one point two litres.
- Step two, the sugar in the second is forty per cent of twelve, four point eight litres.
- Step three, the total sugar is six litres, in a total of twenty litres.
- Six divided by twenty is thirty per cent.
- Check against the trap.
- The plain average of fifteen and forty is twenty seven and a half, but that would only be right for equal amounts.
- The mixture is thirty, nearer the forty, because there is more of the strong syrup.

21 The balance picture

- 15%: 8 L
- Mix: 30%
- 40%: 12 L

IN THE LESSON

- Here is the same mixture drawn on a line, like a seesaw.
- The line runs from weaker on the left to stronger on the right.
- On the left sits the fifteen per cent syrup, with eight litres.
- On the right sits the forty per cent syrup, with twelve litres.
- The mixture lands in between, at thirty per cent.
- Notice where it lands.
- It is fifteen points above the weak syrup, and ten points below the strong one.
- The balance point sits nearer the strong syrup, because there is more of it.
- Like a seesaw, the heavier side is closer to the pivot.
- The distances are in the opposite ratio to the sizes.

22 The balance shortcut

Size A ÷ Size B = Distance to B ÷ Distance to A

Distance to group A	Overall minus A's average
Distance to group B	B's average minus overall
Ratio of sizes	Sizes: opposite to distances

IN THE LESSON

- The shortcut has one rule.
- The distances from the overall average to each group average are in the inverse ratio to the group sizes.
- In words, the group nearer the target is the bigger group.
- Let's name the terms.
- The distance to group A is the gap between the overall average and A's average.
- The distance to group B is the gap to B's average.
- The sizes then split in the opposite order: the size of A over the size of B equals the distance to B over the distance to A.
- In the syrups, the distances were fifteen and ten, so the sizes were in the ratio ten to fifteen, which is two to three.
- That matches eight and twelve.

23 Example 7: finding the sizes by balance

40 students; class averages 70 and 90; overall 76

Distances from 76	6 below, 14 above	
Sizes, inverse ratio	low : high = 14 : 6 = 7 : 3	
One part	40 ÷ 10	**4**
Lower class	7 × 4	**28**
Check	(28×70 + 12×90) ÷ 40	**76**

IN THE LESSON

- Example seven, the typical type.
- A group of forty students is split into two classes.
- One class averages seventy, the other averages ninety, and the whole group averages seventy six.
- How many students are in the lower class?
- Step one, the distances from seventy six are six down to seventy and fourteen up to ninety.
- Step two, the sizes are in the inverse ratio, so the lower class to the upper class is fourteen to six, which is seven to three.
- Step three, seven plus three is ten parts, so each part is four students, and the lower class has twenty eight.
- Check: twenty eight times seventy plus twelve times ninety is three thousand and forty, over forty is seventy six.
- Matching the bigger distance to the bigger group is the trap.

24 Adding water: hold the ingredient fixed

- Water contains none of the ingredient
- Find the ingredient first, then hold it fixed
- New total = ingredient ÷ new strength
- Water added = new total − old total

IN THE LESSON

- Here is a simple story.
- You pour a cup of strong cordial into a jug, then top the jug up with water.
- The jug is now bigger, and weaker.
- But did the amount of cordial change?
- No.
- Only the total volume changed.
- That is the key to every dilution and evaporation question.
- The pure ingredient, the salt, the acid or the sugar, stays the same.
- Water has none of it.
- So work out the amount of the ingredient first, and hold it fixed.
- Then divide that fixed amount by the new strength to get the new total.
- The difference between the new and old totals is how much water you added, or how much boiled away.

25 Example 8: how much water to add?

20 L of 30% salt solution, add pure water to reach 24%

Salt in the start	0.30 × 20	**6 L**
New total at 24%	6 ÷ 0.24	**25 L**
Water added	25 − 20	**5 L**
Check	6 ÷ 25	**24%**
Trap: subtract 6 points	adds 6 litres	**wrong**

IN THE LESSON

- Example eight, the trap type.
- Twenty litres of a thirty per cent salt solution are diluted with pure water until it is twenty four per cent.
- How much water is added?
- Step one, the salt is thirty per cent of twenty, six litres.
- Step two, that six litres stays fixed.
- At twenty four per cent, the new total is six divided by zero point two four, which is twenty five litres.
- Step three, the water added is twenty five minus twenty, five litres.
- A fast check: twenty four per cent is four fifths of thirty per cent, so the volume must grow by a quarter, and a quarter of twenty is five.
- The trap is subtracting six points of strength and adding six litres of water.

26 Replacing and repeating

- Each round keeps the same fraction
- Multiply the remainder by it each time
- Rounds: start × fraction kept ^ rounds
- Do not subtract equal litres

IN THE LESSON

- Another favourite: draw off some liquid and replace it, then do it again.
- Picture a tank of pure juice.
- You take out a quarter of it and top up with water.
- What is left of the juice?
- Three quarters.
- Now you do it again.
- You take out a quarter of the mixture, and that carries away a quarter of the juice that remains.
- So the juice is three quarters, of three quarters.
- Each round multiplies what remains by the same fraction.
- The mistake is to subtract the same amount each time, as if the juice only left in equal litres.
- The amount removed shrinks each round, because the mixture is getting weaker.

27 Example 9: drawing off and replacing

20 L pure fuel; draw off 5 L, replace; repeat once

Fraction kept each round	1 − 5/20	**3/4**
After round one	20 × 3/4	**15 L**
After round two	15 × 3/4	**11.25 L**
Share of the tank	11.25 ÷ 20	**56.25%**
Trap: subtract 5 twice	20 − 10	**10 L**

IN THE LESSON

- Example nine.
- A tank holds twenty litres of pure fuel.
- Five litres are drawn off and replaced with additive.
- Then five litres of the mixture are drawn off and replaced again.
- How much pure fuel is left?
- Step one, each round removes five of twenty, which is a quarter, so three quarters of the fuel stays.
- Step two, after the first round, the fuel is twenty times three quarters, fifteen litres.
- Step three, after the second, fifteen times three quarters is eleven and a quarter litres.
- That is fifty six and a quarter per cent of the tank.
- The trap is to subtract five each time and answer ten litres.
- It would be right only if the second five litres were pure fuel, but they were part additive.

28 Mistakes to avoid in mixtures

- 1. Averages of percents need weights
- 2. Bigger group sits nearer the answer
- 3. Dilution: hold the ingredient fixed
- 4. Replacement: multiply, never subtract
- 5. Add the sizes: total is the denominator

IN THE LESSON

- Mistakes in mixtures and averages.
- One.
- Averaging percents without weights.
- The plain average is right only for equal sizes.
- Two.
- Swapping the weights on the balance.
- The bigger group must sit nearer the overall average, so the bigger distance belongs to the smaller group.
- Three.
- Treating dilution as subtracting points of strength.
- Hold the ingredient fixed and divide.
- Four.
- Subtracting the same litres at each replacement.
- Multiply by the fraction that stays.
- Five.
- Forgetting to add the sizes at the end.
- The total is the bottom of the fraction, and it is the combined size.
- A quick sense check always helps: the result must sit between the two group averages.

29 Recap: five rules for both skills

Two skills, five rules

Name the base	Percent of what? Cost or tag
Use multipliers	Chain them, divide to go back
Interest rule	Prt, or multiply per period
Weighted mix	Total ingredient ÷ total size
Hold one thing fixed	Water adds no ingredient

IN THE LESSON

- Let's pull it together.
- First, name the base: profit and markup sit on cost, discount on the marked price.
- Second, chain multipliers, and divide by the chain to go backwards.
- Third, interest: simple is principal times rate times years, compound multiplies by one plus r over n, once for each period.
- Fourth, a mixture is a weighted average, total ingredient over total size, and on the balance the distances run opposite to the sizes.
- Fifth, in dilution or evaporation, hold the ingredient fixed, and in repeated replacement, multiply by the fraction that stays.
- With these five, the two families of questions become routine.

30 Try three quick questions

1. Cost $80, sold at $100. Profit as a percent of cost?
   Answer: 20 ÷ 80 = 25%.
2. $400 at 10% a year, compounded yearly. Balance after 2 years?
   Answer: 400 × 1.1 × 1.1 = $484.
3. 10 L of 20% acid + 10 L of 60% acid. Strength?
   Answer: Equal sizes: plain average, 40%.

IN THE LESSON

- Now three quick questions.
- Pause after each one.
- Question one.
- A seller buys an item for eighty dollars and sells it for one hundred.
- Profit is what percent of cost?
- Profit is twenty dollars, over the cost of eighty, which is twenty five per cent.
- Question two.
- Four hundred dollars is invested at ten per cent a year, compounded once a year.
- What is the balance after two years?
- One point one times one point one is one point two one, and four hundred times one point two one is four hundred eighty four dollars.
- Question three.
- Ten litres of twenty per cent acid are mixed with ten litres of sixty per cent acid.
- What is the strength?
- The sizes are equal, so the answer is the plain average, forty per cent.
- Well done.