Counting and probability

Quantitative Reasoning

1 A story: the sandwich shop

- 4 breads, 3 fillings, 5 sides: how many lunches?
- Do not list them. Build the choice step by step
- Options at each step multiply
- This one idea is the base of all counting

IN THE LESSON

- Picture a sandwich shop with a simple menu.
- You choose one bread from four, one filling from three, and one side from five.
- How many different lunches are there?
- You could list them all, but you would be there all afternoon.
- Instead, think of the choices as a chain.
- Pick the bread: four ways.
- For each bread, pick a filling: three ways.
- For each of those, pick a side: five ways.
- The number of lunches is what you get by multiplying along the chain.
- That single idea, multiply the number of options at each step, is the foundation of all counting on the exam.

2 The multiplication principle

Total = m × n × ...

First choice: m ways	Options for the first step
Second choice: n ways	Options for the second step
Total ways	Multiply the options

IN THE LESSON

- Here is the rule.
- If one choice can be made in m ways, and a second choice can be made in n ways, then the two choices together can be made in m times n ways.
- The first term is the number of options for the first choice.
- The second term is the number of options for the second choice.
- The product is the total number of combined outcomes.
- It works for any number of steps, you simply keep multiplying.
- One condition: the number of options at the second step must not depend on which option you took first, or else you use the number that is left each time, as we will see in a moment.

3 Example 1: lunches and codes

Multiply the options at each step

Lunches: 4 × 3 × 5	bread, filling, side	**60**
Code, repeats allowed	10 × 10 × 10 × 10	**10,000**
Code, all different	10 × 9 × 8 × 7	**5,040**

No reuse: the options shrink by one each step

IN THE LESSON

- Example one.
- First, the shop.
- Four breads, times three fillings, times five sides gives sixty lunches.
- Now a lock code.
- A code has four digits, and each digit can be any of zero to nine.
- If digits may repeat, every place has ten options, so ten times ten times ten times ten, ten thousand codes.
- Now say the digits must all be different.
- The first place has ten options, but the second has only nine left, then eight, then seven.
- Ten times nine times eight times seven is five thousand and forty.
- Spot the pattern: when you cannot reuse something, the options shrink by one each step.

4 The big decision: does order matter?

Order matters: roles, ranks, rows, codes

Fill the seats one at a time →
Multiply the shrinking options → Permutation

Order ignored: committee, team, hand

Count it as if order mattered →
Divide by the orderings of the group → Combination

IN THE LESSON

- Now the most important decision in counting.
- Before you calculate anything, ask: if I swap two of the chosen things, do I get a different outcome?
- Think of picking a president and a treasurer from a group.
- Swapping them gives a different result, because the jobs are different.
- So order matters, and we count ordered picks, called permutations.
- Now think of picking two people for a committee of equal members.
- Swapping them changes nothing.
- Order does not matter, so we count groups, called combinations.
- Read the question for the clue words.
- Roles, ranks, positions and passwords mean order matters.
- Committee, team, hand and subset mean it does not.

5 Factorials: lining things up

n! = n × (n − 1) × (n − 2) × ... × 1

n factorial	Multiply a countdown
Count down to one	Stop at 1; 0! is 1
Ways to line up n things	5 books: 5! = 120

IN THE LESSON

- A shortcut notation first.
- Five factorial, written with an exclamation mark, means five times four times three times two times one.
- In general, n factorial is n times n minus one, and so on, down to one.
- Why does it count arrangements?
- To line up n different things in a row, the first place has n choices, the second has n minus one, and so on until one is left.
- That product is exactly n factorial.
- Five books on a shelf can be ordered in one hundred and twenty ways.
- Two facts to remember: zero factorial is defined to be one, and factorials grow very fast, so cancel before you multiply.

6 Permutations: ordered picks

P(n, k) = n! ÷ (n − k)! = n × (n − 1) × ... (k numbers)

Things to pick from: n	Total available
Seats to fill: k	How many positions
Ordered picks	k numbers counting down

IN THE LESSON

- Now suppose you do not use every item, only k of the n.
- A club has eight members and must choose a president, a treasurer and a secretary.
- The first seat has eight options, the second seven, the third six.
- That is eight times seven times six, which is three hundred and thirty six.
- The formula says the same thing, n factorial divided by n minus k factorial, because dividing wipes out the tail of the countdown you did not use.
- In practice, never use the formula.
- Just write k numbers counting down from n, and multiply.
- Three seats means three numbers.

7 Combinations: groups

C(n, k) = n! ÷ (k! × (n − k)!) = P(n, k) ÷ k!

Things to pick from: n	Total available
Group size: k	Members in each group
Groups, order ignored	Ordered count over k!

IN THE LESSON

- For a committee, order does not matter.
- Count the ordered picks first, then fix the overcount.
- Every group of three people appears in the ordered list in six different orders, since three people can be lined up in three factorial ways, which is six.
- So divide by six.
- The formula is n factorial over k factorial times n minus k factorial, often read as n choose k.
- Again, in practice: write k numbers counting down from n on top, k numbers counting down from k on the bottom, and cancel.
- Combinations are always smaller than permutations of the same numbers, unless k is one.

8 Example 2: officers or committee

Pick 3 people from 8

Three different jobs	8 × 7 × 6	**336**
Orders of 3 people	3 × 2 × 1	**6**
Equal committee	336 ÷ 6	**56**

Both numbers appear among the answer choices

IN THE LESSON

- Example two.
- Eight people, three chosen.
- First case: a chair, a deputy and a clerk, three different jobs.
- Order matters, so it is eight times seven times six, three hundred and thirty six.
- Second case: a committee of three equal members.
- Order does not matter.
- Take the three hundred and thirty six ordered picks and divide by six, the number of ways to order three people.
- That gives fifty six.
- Do the cancelling at the start: eight times seven times six over three times two times one.
- The six on the bottom cancels with the six on top, leaving eight times seven, fifty six.
- The exam offers both answers, so decide the order question first.

9 Handy values to know by heart

Pairs from a group: each pair counted once

People in the room	Pairs
4 people	6
6 people	15
8 people	28
10 people	45

IN THE LESSON

- Some combination values appear again and again, so learn a few.
- Choosing two from a group is the handshake count: in a room of n people, each pair shakes hands once.
- The count is n times n minus one over two.
- Four people give six handshakes, six give fifteen, eight give twenty eight, and ten give forty five.
- There is also a symmetry: choosing the people to include is the same as choosing the people to leave out.
- So eight choose three equals eight choose five, and ten choose eight equals ten choose two.
- When k is more than half of n, switch to the smaller number, and the arithmetic gets much easier.

10 Example 3: at least one, by subtraction

3 chosen from 4 men and 5 women, at least 1 woman

All committees	9 choose 3	**84**
No woman: all men	4 choose 3	**4**
At least one woman	84 − 4	**80**

Check by cases: 30 + 40 + 10 = 80

IN THE LESSON

- Example three, a typical exam restriction.
- A committee of three is chosen from four men and five women.
- It must include at least one woman.
- How many committees?
- You could split into cases, one woman, two women, three women, and add.
- That works, and gives thirty plus forty plus ten.
- But look at the shortcut.
- Count all committees of three from nine people, which is eighty four.
- Now subtract the committees that break the rule, those with no woman at all, so three men from four, which is four.
- Eighty four minus four is eighty.
- The words at least one are a signal: count everything, then subtract the cases that fail.

11 Example 4: people who must sit together

5 friends in a row; Asha and Ben together

Glue them: 4 things	4!	**24**
Swap inside the block	× 2	**48**
All arrangements	5!	**120**
Not together	120 − 48	**72**

IN THE LESSON

- Example four, the glue trick.
- Five friends sit in a row of five seats.
- Two of them, Asha and Ben, insist on sitting next to each other.
- How many arrangements?
- Treat Asha and Ben as one glued block.
- Now you are arranging four things, the block and three others, which is four factorial, twenty four.
- But inside the block, Asha and Ben can swap, so multiply by two.
- That gives forty eight.
- And if the question asks for the two to be apart?
- Take the total of one hundred and twenty and subtract the forty eight where they are together.
- The answer is seventy two.
- Again, count what is easy, subtract what breaks the rule.

12 Example 5: letters that repeat

Arrangements of the letters in BOOK

If all four were different	4!	**24**
The two O's are identical	2!	**2**
Different words	24 ÷ 2	**12**

Divide by the factorial of each repeated count

IN THE LESSON

- Example five, a trap built into the question.
- How many different arrangements are there of the letters in the word BOOK?
- Four letters, so you might say four factorial, twenty four.
- But the two O's are identical, so swapping them gives the same word, and every arrangement has been counted twice.
- Divide by two factorial, which is two.
- Twenty four over two is twelve.
- The general rule is to take the factorial of the total number of letters, and divide by the factorial of each repeated count.
- A word with two repeated pairs divides by two factorials.
- So check the word for repeats before you start.

13 Counting mistakes to avoid

- Ordered count used when order does not matter
- Identical items: divide by their factorials
- And means multiply. Separate cases mean add
- Overlapping cases double count. Cancel before multiplying
- Restriction? Total minus the bad cases

IN THE LESSON

- Here are the counting mistakes that lose the most marks.
- One.
- Using the ordered count when order does not matter, or the other way round.
- Always ask the swap question first.
- Two.
- Forgetting that identical items, like the two O's, make some arrangements the same.
- Three.
- Adding when you should multiply.
- If you do one thing and then another, multiply.
- If you do one thing or another, separate and exclusive cases, add.
- Four.
- Splitting into cases that overlap, so some outcomes are counted twice.
- Also, working out a huge factorial in full.
- Always cancel first, and use the smaller of k and n minus k.
- Five.
- For restrictions, try subtracting the bad cases from the total.

14 Next: from counting to chance

- Bag of 10 sweets, 3 red: chance of red is 3/10
- Probability = share of equally likely outcomes
- Zero: impossible. One: certain
- Counting gives the top and bottom of the fraction

IN THE LESSON

- Now we turn to probability.
- Think of a bag of ten sweets, three of them red.
- You close your eyes and take one.
- How likely is red?
- Three out of ten.
- That is the whole idea.
- When every outcome is equally likely, probability is simply a share of the outcomes: the ones you want, out of all the ones that could happen.
- A probability is a number from zero to one, where zero means impossible and one means certain.
- You can write it as a fraction, a decimal or a percentage.
- On the exam, fractions are usually best, because they stay exact and they cancel nicely.
- And the counting you just learned is exactly how you find the two numbers.

15 Probability as a ratio of counts

P(event) = favourable outcomes ÷ all equally likely outcomes

Favourable outcomes	The ones you want
All equally likely outcomes	Counted the same way
Probability	A number from 0 to 1

IN THE LESSON

- The formula is a fraction.
- Probability equals the number of favourable outcomes, the ones you want, divided by the total number of equally likely outcomes.
- Take each part.
- The top is how many outcomes satisfy the question.
- The bottom is how many outcomes are possible in all.
- The two counts must be measured the same way.
- If the top counts ordered outcomes, the bottom must count ordered outcomes too.
- If the top counts groups, the bottom counts groups.
- This is the most common source of wrong answers.
- Mixing an ordered top with an unordered bottom gives a fraction that looks sensible and is quietly wrong.

16 Example 6: two dice

Total of two fair dice equals 7

All ordered outcomes	6 × 6	**36**
Pairs that make 7	1+6, 2+5, 3+4 ...	**6**
Probability	6 ÷ 36	**1/6**

IN THE LESSON

- Example six.
- Roll two fair dice, and ask for the chance that the total is seven.
- First the bottom number.
- The first die has six results, the second has six, so there are six times six, thirty six equally likely ordered outcomes.
- Treat the dice as different, say one red and one blue, even if they look the same.
- Now the top number.
- Which pairs add to seven?
- One and six, two and five, three and four, four and three, five and two, six and one.
- That is six pairs.
- So the chance is six over thirty six, which simplifies to one sixth.
- Notice that one and six and six and one are different outcomes, so both count.

17 Why seven is the favourite total

Ways, out of 36

Total 2	**1**
Total 3	**2**
Total 5	**4**
Total 7	**6**
Total 9	**4**
Total 11	**2**

Outcomes are pairs, not totals

IN THE LESSON

- Here is a picture of the dice.
- Each bar is a possible total, and its height is the number of ways to make it, out of thirty six.
- Start at the left.
- A total of two can happen only one way, a one and a one.
- Three can happen two ways.
- Five can happen four ways.
- Seven peaks at six ways.
- Then it falls again: nine has four ways, and eleven has two.
- The totals are not equally likely, which is why we count the ordered pairs rather than just the totals.
- If you counted eleven totals and called each one in eleven, you would be wrong.
- All eleven bars would add to thirty six.

18 Or, and, and not

P(A or B) = P(A) + P(B) − P(A and B)
Independent: P(A and B) = P(A) × P(B)
Not A: P(not A) = 1 − P(A)

P(A or B)	Add, then remove the overlap
Independent events	Multiply the separate chances
Complement rule	One minus the chance

IN THE LESSON

- Three rules cover almost everything.
- First, for the chance that A or B happens, add the two chances and subtract the chance that both happen, because the overlap was counted twice.
- If the events cannot happen together, there is no overlap, and you just add.
- Second, for the chance that two independent events both happen, multiply.
- Independent means that one result does not change the chance of the other, like two separate coin tosses.
- Third, the complement rule.
- The chance that A does not happen is one minus the chance that it does.
- Everything adds up to one, so the leftover is what you want.

19 Example 7: one die, or the overlap

One fair die: even, or greater than 4

P(even)	2, 4, 6	**3/6**
P(greater than 4)	5, 6	**2/6**
Overlap: P(both)	only 6	**1/6**
P(even or greater than 4)	3/6 + 2/6 − 1/6	**2/3**

Adding alone gives 5/6: the six counted twice

IN THE LESSON

- Example seven.
- Roll a single fair die.
- What is the chance of an even number, or a number greater than four?
- Even numbers are two, four and six, so three out of six.
- Numbers greater than four are five and six, two out of six.
- If you just add, you get five sixths.
- But six is in both lists, so it was counted twice.
- The overlap is one sixth.
- So subtract it: three sixths plus two sixths minus one sixth is four sixths, which is two thirds.
- Check by listing: the outcomes two, four, five and six qualify, four of six.
- A quick list for small problems is the best way to confirm a formula.

20 Example 8: independent events

Coin gives heads; die gives 5 or 6

P(heads)		**1/2**
P(5 or 6)	2 ÷ 6	**1/3**
Both: multiply	1/2 × 1/3	**1/6**
Check by counting	2 ÷ 12	**1/6**

IN THE LESSON

- Example eight.
- Toss a fair coin and roll a fair die.
- What is the chance of heads, and a five or a six?
- The coin and the die do not affect each other, so the events are independent, and we multiply.
- The coin gives heads with chance one half.
- The die gives a five or a six with chance two sixths, which is one third.
- Multiply: one half times one third is one sixth.
- Cross-check by counting: twelve equally likely outcomes, two coin results times six die results.
- The wanted ones are heads with five, and heads with six.
- Two of twelve, again one sixth.
- Remember the word and means multiply, but only if the events are independent.

21 Example 9: draws without replacement

Bag: 5 red, 3 green, 2 blue. Draw 2, no replacement

1st draw red	5 of 10	**5/10**
2nd draw red, given first	4 of 9 left	**4/9**
Both red	5/10 × 4/9	**2/9**
By combinations	C(5,2) ÷ C(10,2)	**10/45**

Both routes give 2/9

IN THE LESSON

- Example nine, the exam's favourite twist.
- A bag holds five red, three green and two blue marbles, ten in all.
- You draw two, one after the other, without putting the first back.
- What is the chance both are red?
- The first draw is red with chance five tenths.
- Now the bag has changed.
- Only nine marbles are left, and only four are red.
- So the second chance is four ninths, not five tenths again.
- Multiply: five tenths times four ninths is twenty ninetieths, which is two ninths.
- There is a second route, with combinations: ways to choose two reds from five, ten, over ways to choose two marbles from ten, forty five.
- Ten over forty five is two ninths again.

22 At least one: the one minus none trick

At least one = 1 − none

Chance one try fails →

Multiply for every try: none → Subtract from 1

IN THE LESSON

- Now the shortcut that wins the most time on the exam.
- A question asks for the chance of at least one success.
- That covers many different ways: exactly one, exactly two, and so on up to all of them.
- Adding all of those up is slow and risky.
- But there is only one way to get none.
- So work out the chance of none, and take it away from one.
- Three steps.
- Step one, find the chance of a failure on a single try.
- Step two, raise it to the power of the number of tries, or multiply the failures together, for none.
- Step three, subtract the result from one.
- Whenever you see the words at least one, say them aloud as one minus none.

23 Example 10: three attempts

Prize chance 1/5 per attempt, 3 independent attempts

One attempt fails	1 − 1/5	**4/5**
All three fail	$(4/5)^3$	**64/125**
At least one win	1 − 64/125	**61/125**

Adding 1/5 three times would give 3/5: wrong

IN THE LESSON

- Example ten.
- A shop gives out a prize with chance one in five on each attempt, and a player makes three independent attempts.
- What is the chance of winning at least once?
- Do not add one fifth three times, because that would give three fifths, and the true answer is not that.
- Instead, one try fails with chance four fifths.
- Three failures in a row: four fifths times four fifths times four fifths, which is sixty four over one hundred and twenty five.
- At least one win is one minus that, which is sixty one over one hundred and twenty five.
- Just under half.
- Sense check: it must be more than one fifth, and it is.

24 Example 11: exactly two heads

Fair coin, 3 tosses, exactly 2 heads

Ways to place 2 heads	3 choose 2	**3**
One sequence, e.g. HHT	$(1/2)^3$	**1/8**
Exactly two heads	3 × 1/8	**3/8**

Check: 1 + 3 + 3 + 1 = 8 sequences in all

IN THE LESSON

- Example eleven, the other kind of question.
- Toss a fair coin three times.
- What is the chance of exactly two heads?
- Here you cannot use the complement, so you count cases and add.
- List the ways: head, head, tail.
- Head, tail, head.
- Tail, head, head.
- That is three ways, and you can also get it from combinations: choose which two of the three tosses are heads, three choose two, which is three.
- Each particular sequence has chance one half cubed, one eighth.
- Three sequences, so three eighths.
- The method for exactly k successes is to count the positions with a combination, and multiply by the chance of one sequence.
- All together, that is every outcome: one, three, three, one, out of eight.

25 Probability traps

- Add chances only for exclusive events, or subtract overlap
- No replacement: the pool and the chance change
- Top and bottom counted the same way
- At least one: use 1 − none, never add
- An answer above 1 is always wrong

IN THE LESSON

- Here are the traps that appear again and again in probability questions.
- One.
- Adding the chances of events that can happen together.
- Always subtract the overlap, or check that the events are exclusive.
- Two.
- Multiplying chances when the events are not independent, and especially, using the same chance for the second draw when nothing was put back.
- Three.
- Counting ordered pairs on the top and unordered pairs on the bottom, or the other way round.
- Four.
- Adding the single chances for at least one, instead of one minus none.
- Five.
- Giving an answer larger than one.
- A probability is never more than one, so check every answer against that.

26 Recap: counting and chance

P(n,k) = k numbers down from n C(n,k) = P(n,k) ÷ k!
P(A and B) = P(A) × P(B) P(A or B) = P(A) + P(B) − P(A and B)
At least one = 1 − P(none)

Order matters: P(n,k)	Count down k numbers
Order ignored: C(n,k)	Divide by k factorial
Multiply for and	Independent: multiply
Subtract the overlap	Or: add, remove overlap
One minus none	At least one: one minus none

IN THE LESSON

- Let's pull it together.
- If order matters, count the shrinking options, k numbers counting down from n.
- If order does not matter, divide those by k factorial.
- For and, multiply, with the later chance adjusted if nothing is replaced.
- For or, add and subtract the overlap.
- For at least one, work out one minus none.
- And remember the first question every time, which is whether the swap changes the outcome.
- With the signal words, the right method usually jumps out.
- Roles and rows are permutations.
- Committees and hands are combinations.
- At least one is one minus none.

27 Try three quick questions

1. Choose a committee of 3 from 6 people. How many?
   Answer: 6 × 5 × 4 ÷ 6 = 20
2. Bag: 2 red, 3 blue. Draw 2, no replacement. P(both blue)?
   Answer: 3/5 × 2/4 = 3/10
3. A die is rolled twice. P(at least one six)?
   Answer: 1 − (5/6)² = 11/36

IN THE LESSON

- Now try three questions.
- Pause after each one.
- Question one.
- A committee of three is chosen from six people.
- How many committees?
- Order does not matter.
- Six times five times four over three times two times one is one hundred and twenty over six, which is twenty.
- Question two.
- A bag holds two red and three blue marbles.
- Two are drawn without replacement.
- What is the chance both are blue?
- Three fifths times two quarters is six twentieths, three tenths.
- Question three.
- A fair die is rolled twice.
- What is the chance of at least one six?
- One minus five sixths squared is one minus twenty five thirty sixths, which is eleven thirty sixths.