CHARTA LEARNING · LESSON NOTES

Estimation and answer elimination; sequences and series

GMAT Quantitative Reasoning

1 A story: the restaurant bill

- Bill $187.60 for 6 people
- Round: about $200 ÷ 6 ≈ $33
- Rough is enough if choices are far apart
- No calculator, about two minutes a question

IN THE LESSON

- Picture a dinner with friends.
- The bill comes to one hundred and eighty seven dollars and sixty cents, and there are six of you.
- Nobody reaches for a calculator.
- Someone says, about two hundred, split six ways, about thirty three each.
- And that is good enough, because the choices in front of you are ten, twenty, thirty, forty and fifty, you only need to be close.
- Well, on the exam it is the same game.
- There is no calculator, and about two minutes a question.
- When the five answer choices are far apart, a rough answer picks the right one, and you never do the exact sum.

2 The estimation method in four moves

Estimate, then eliminate

Scan the choices: how far apart? →
Round to easy numbers →
Track which way each moved →
Compute and compare

IN THE LESSON

- Here is the method as four moves.
- Move one, look at the answer choices before you touch the numbers.
- How far apart are they?
- That tells you how rough you can be.
- Move two, round every number to something easy, such as a nice whole number, or a power of ten times a single digit.
- Move three, and this is the one people skip, note the direction of each rounding.
- Did you round up or down, and does that push the answer up or down?
- Move four, do the easy arithmetic and compare with the choices.
- The question will often say closest to or approximately.
- That wording is the exam inviting you to estimate.

3 Example 1: a plain estimate

Closest to (31.8 × 9.9) ÷ 0.51 ?

Round	32 × 10 ÷ 0.5	
Top part	32 × 10	**320**
Divide by 0.5	320 × 2	**640**
Exact, to check	314.82 ÷ 0.51	**about 617**

Choices: 64, 160, 320, 640, 1280. Pick 640.

IN THE LESSON

- Example one.
- Which of these is closest to thirty one point eight times nine point nine, divided by nought point five one?
- The choices are sixty four, one hundred and sixty, three hundred and twenty, six hundred and forty, and twelve hundred and eighty.
- Notice they double each time, so we can be very rough.
- Round thirty one point eight to thirty two.
- Round nine point nine to ten.
- Round nought point five one to nought point five.
- Now thirty two times ten is three hundred and twenty.
- Divide by one half, which is the same as doubling, and we get six hundred and forty.
- The exact value is about six hundred and seventeen, so six hundred and forty is clearly the closest.

4 Decimals: where the point goes

- 0.004 × 0.03 = 0.00012 (5 places in all)
- Divide by a number below 1: answer grows
- 45 ÷ 0.09 = 500; ÷ 0.5 doubles
- Always ask: bigger or smaller than before?

IN THE LESSON

- Estimation fails most often on the decimal point, so let us fix that.
- When you multiply decimals, count the places.
- Nought point nought nought four times nought point nought three: the digits are four times three, twelve, and there are three places plus two places, five in all, so the answer is nought point nought nought nought one two.
- Now division.
- Dividing by a number between nought and one makes the answer bigger, not smaller.
- Forty five divided by nought point nought nine is five hundred, because nought point nought nine is about a tenth.
- Dividing by one half doubles.
- A quick check at the end: ask whether your answer should be bigger or smaller than where you started.

5 Boxing in a long sum

n × smallest ≤ S ≤ n × largest

How many terms (n)	Count of terms added
Smallest term	The lowest term
Largest term	The highest term
The sum (S)	The sum, boxed in

IN THE LESSON

- Sometimes there is nothing to round, because the sum is long.
- The trick is to box it in.
- Here is the rule.
- Every term in the sum is at least as big as the smallest term, and at most as big as the largest.
- So if you have n terms, the sum is at least n times the smallest, and at most n times the largest.
- Let us take the terms in turn.
- The letter n is how many terms you are adding.
- The smallest term is the lowest value in the list, and the largest term is the biggest.
- The sum, written with a capital S, lives somewhere between the two products.
- That alone is often enough to remove four choices.

6 Example 2: a sum nobody should add

S = 1/21 + 1/22 + … + 1/30. Closest to?

Number of terms	30 − 21 + 1	**10**
Low bound	10 × 1/30	**0.333**
High bound	10 × 1/21	**0.476**
So S is between	0.333 and 0.476	**pick 0.4**

Choices: 0.2, 0.4, 0.6, 0.8, 1.2

IN THE LESSON

- Example two.
- Let S be the sum of one over twenty one, plus one over twenty two, and so on, up to one over thirty.
- Which of these is closest to S: nought point two, nought point four, nought point six, nought point eight, or one point two?
- Count the terms.
- From twenty one to thirty is ten terms.
- The smallest term is one over thirty, the largest is one over twenty one.
- Ten times one over thirty is one third, about nought point three three.
- Ten times one over twenty one is a little under nought point four eight.
- So S lies between nought point three three and nought point four eight, and only nought point four fits.
- No fractions were added at all.

7 The sum sits between its two bounds

Value of the sum S

Low bound	**0.33**
True sum	**0.40**
High bound	**0.48**

True sum is about 0.40, between the bounds

IN THE LESSON

- Here is that last example as a picture.
- The height of each bar is a value.
- Read the vertical axis first: it shows the size of the sum.
- The left bar is the low bound, ten times the smallest term, about nought point three three.
- The right bar is the high bound, ten times the largest term, about nought point four eight.
- The middle bar is the true sum, which a computer says is about nought point four.
- It sits between the others, exactly as the rule promised.
- Notice the true value is not always in the exact middle, it is closer to the low bound here, because most terms are nearer to one over thirty.
- But you never needed it.
- The box was enough.

8 Comparing fractions near one half

- 17/35 is 1/70 below 1/2
- 19/39 is 1/78 below 1/2
- Smaller gap below a half means bigger fraction
- So 19/39 > 17/35. No cross multiplying

IN THE LESSON

- Here is a shortcut for comparing fractions that sit close to one half.
- Do not cross multiply, instead measure the distance from one half.
- Take seventeen over thirty five.
- Half of thirty five is seventeen and a half, so seventeen is half a unit short.
- The fraction is below one half by half over thirty five, which is one seventieth.
- Now nineteen over thirty nine.
- Half of thirty nine is nineteen and a half, again half a unit short.
- It is below one half by one seventy eighth.
- One seventy eighth is smaller than one seventieth, so nineteen over thirty nine is closer to one half, and it is the bigger fraction.
- Same idea works for fractions near one, or near any easy number.

9 Example 3: the trap, do not round away the point

Closest to $20.3^2 - 19.7^2$?

Tempting but wrong	$20^2 - 20^2$	**0**
a − b	20.3 − 19.7	**0.6**
a + b	20.3 + 19.7	**40**
Product	0.6 × 40	**24**

Choices: 0, 12, 24, 40, 400. Pick 24.

IN THE LESSON

- Example three is the trap.
- Which of these is closest to twenty point three squared, minus nineteen point seven squared?
- The choices are zero, twelve, twenty four, forty and four hundred.
- Tempting move: round both to twenty.
- Twenty squared minus twenty squared is zero, and choice A looks great.
- But the question is about the difference, and you just rounded the difference away.
- Instead use the identity that a squared minus b squared equals a minus b, times a plus b.
- The first bracket is nought point six.
- The second is forty.
- Nought point six times forty is twenty four.
- Round only what is safe, and never the thing the question asks about.

10 Mistakes to avoid when estimating

- Do not round away a small difference
- Track the direction of every rounding
- Count decimal places carefully
- Choices close together? Be less rough
- Says exactly? Do the exact work

IN THE LESSON

- Let us pause and collect the traps so far.
- First, rounding away the very thing the question is about, as in the difference of squares.
- When the answer is a small gap between two big numbers, estimation by rounding is dangerous, so use algebra.
- Second, losing track of direction.
- If you round the top of a fraction up and the bottom down, your estimate is too high.
- Knowing that can break a tie between two close choices.
- Third, decimal places, which we fixed earlier.
- Fourth, rounding too roughly when the choices are close together, such as sixty two and sixty three.
- Then estimate less, or use the last digit.
- Finally, estimating when the question says exactly.
- Then do the exact work.

11 Example 4: percent changes in a row

$80.20, up 29.8%, then down 20.1%. Closest?

Round	$80, +30%, −20%	
After the rise	80 × 1.3	**$104**
After the cut	104 × 0.8	**$83.2**
Exact, to check	80.20 × 1.298 × 0.799	**$83.18**

Choices: 64, 83, 104, 112, 128. Pick 83.

IN THE LESSON

- Example four.
- A jacket costs eighty dollars and twenty cents.
- The price is raised by twenty nine point eight per cent, then the new price is cut by twenty point one per cent.
- Which is closest to the final price?
- The choices are sixty four, eighty three, one hundred and four, one hundred and twelve, and one hundred and twenty eight dollars.
- Round the price to eighty, the rise to thirty per cent and the cut to twenty.
- Raising by thirty per cent means multiply by one point three, giving one hundred and four.
- Cutting by twenty per cent means multiply by nought point eight, giving eighty three point two.
- The final price is closest to eighty three dollars.
- Notice a rise followed by an equal cut does not return home.

12 Eliminating when the choices are close

- Last digit: 7 × 8 ends in 6
- 37 × 48 = 1,776, the only choice ending in 6
- Check odd or even, and sign, and size
- Use the quickest fact that cuts the choices

IN THE LESSON

- What if the choices are too close for rounding?
- Then use other facts about the answer to cross choices out.
- One is the last digit.
- Take thirty seven times forty eight.
- The last digits are seven and eight, and seven times eight is fifty six, so the product must end in six.
- If the choices are seventeen seventy one, seventeen seventy three, seventeen seventy six, seventeen seventy eight and seventeen seventy nine, only one ends in six.
- Done.
- Another is size.
- Forty times fifty is two thousand, but we rounded both numbers up, so two thousand is too high.
- A third is odd or even.
- An even number times anything is even.
- Use whatever fact is fastest.

13 Sequences: a list with a rule

- Sequence: numbers in order, built by a rule
- Term: one number in the list
- Series: the terms added together
- Asked: a term, a sum, or how many terms

IN THE LESSON

- Now our second skill.
- Imagine a theatre.
- The first row has twelve seats, and every row behind it has five more seats than the row in front.
- Row one has twelve, row two seventeen, row three twenty two.
- That list of numbers, in order, is a sequence, and each number is called a term.
- The rule that builds it is what matters.
- Here the rule is add five each time.
- A series is what you get when you add the terms up, for instance the total seats in the first ten rows.
- The exam asks three things about them.
- What is a particular term?
- What is the sum of a stretch of terms?
- And how many terms are there?
- Let us take each in turn.

14 Arithmetic sequences and the fence post

- Arithmetic: add the same d each time
- 10 posts have 9 gaps
- Steps to reach term n: n − 1
- Items from 22 to 60: 60 − 22 + 1 = 39

IN THE LESSON

- The theatre is an arithmetic sequence, one where each term adds the same amount, called the common difference, d.
- To reach the tenth row, how many times do you add five?
- Not ten.
- You start on row one, so you take nine steps.
- This is the fence post problem.
- A fence with ten posts has only nine gaps between them.
- The term number counts posts, the difference is added once for each gap, so it is added n minus one times.
- Counting how many integers lie from twenty two to sixty is the same trap.
- Sixty minus twenty two is thirty eight gaps, so there are thirty nine posts.
- Always add one when you count items, and subtract one when you count steps.

15 The nth term of an arithmetic sequence

$a_n = a_1 + (n - 1) \times d$

First term (a1)	Where the list starts
Common difference (d)	Added at every step
Term number n	Which position you want
Term number n (an)	The term you are after

IN THE LESSON

- Here is the formula.
- The nth term equals the first term, plus n minus one, times the common difference.
- Take each part in turn.
- The first term, a one, is where you start.
- The common difference, d, is what you add each step, and it can be negative if the list goes down.
- The letter n is the position you are after, first, second, thirtieth.
- The answer, a n, is the term in that position.
- The n minus one is the fence post: the number of steps from the first term.
- Check it on the theatre.
- Row three should be twelve plus two times five, which is twenty two.
- Matches the list.

16 Example 5: finding the thirtieth term

12, 17, 22, … Find the 30th term

Difference d	17 − 12	**5**
Steps	30 − 1	**29**
Jump from start	29 × 5	**145**
30th term	12 + 145	**157**

Trap: 12 + 30 × 5 = 162, one step too many

IN THE LESSON

- Example five.
- A sequence starts twelve, seventeen, twenty two, and goes on in the same way.
- What is the thirtieth term?
- The choices are one hundred and fifty, one hundred and fifty two, one hundred and fifty seven, one hundred and sixty two, and one hundred and sixty seven.
- First, find the difference: seventeen minus twelve is five.
- Second, count the steps: thirty minus one is twenty nine.
- Third, add twenty nine fives to the start.
- Twenty nine times five is one hundred and forty five.
- Add twelve and the answer is one hundred and fifty seven.
- The trap choice is one hundred and sixty two, which comes from using thirty steps instead of twenty nine.

17 Adding fast: the pairing trick

- 1 + 100 = 101, 2 + 99 = 101, …
- 50 pairs, each worth 101
- Total = 50 × 101 = 5,050
- Terms × average of first and last

IN THE LESSON

- How do you add a long run of terms?
- A story about a schoolboy called Gauss.
- His teacher asked the class to add every whole number from one to one hundred, hoping for a quiet hour.
- Gauss wrote the answer in seconds.
- He paired the first and last, one plus one hundred, which is one hundred and one.
- Then the second and second last, two plus ninety nine, again one hundred and one.
- Every pair gives one hundred and one, and there are fifty pairs.
- So the total is fifty times one hundred and one, which is five thousand and fifty.
- Fifty pairs is half of one hundred terms.
- So the sum is the number of terms, times the average of the first and last.

18 The sum of an arithmetic series

S = n × (first + last) ÷ 2

How many terms (n)	How many terms
First term (a1)	First term of the run
Last term	Last term of the run
The sum (S)	The total

IN THE LESSON

- Here is the formula.
- The sum equals the number of terms, times the first term plus the last term, divided by two.
- Look at the parts.
- The letter n is the number of terms, found with the fence post rule: last minus first, divided by the difference, plus one.
- The first and last terms are the two ends of the run.
- Adding them and halving gives the average of the two ends, which, in an arithmetic list, is also the average of all the terms.
- Multiply that average by how many terms there are, and you have the total.
- This works for any arithmetic run, whether it counts by ones, threes or sevens.

19 Example 6: summing multiples of three

Sum of multiples of 3 from 15 to 99

Steps of 3	(99 − 15) ÷ 3	**28**
Terms	28 + 1	**29**
Average of ends	(15 + 99) ÷ 2	**57**
Sum	29 × 57	**1,653**

Check: about 60 × 29 ≈ 1,700

IN THE LESSON

- Example six.
- What is the sum of all the multiples of three from fifteen to ninety nine, inclusive?
- The choices are sixteen hundred and fifty three, seventeen hundred and thirty two, eighteen hundred, nineteen hundred and eighty, and two thousand and sixteen.
- Step one, count the terms.
- Ninety nine minus fifteen is eighty four.
- Divided by three, that is twenty eight steps, so twenty nine terms.
- Step two, average the ends: fifteen plus ninety nine is one hundred and fourteen, halved is fifty seven.
- Step three, multiply, twenty nine times fifty seven, which is sixteen hundred and fifty three.
- As a check, the average is about sixty, and twenty nine terms gives about seventeen hundred, so sixteen fifty three is sensible.

20 Counting terms: the fence post table

Count the items, not the gaps

Question	Count
Integers 41 to 90	90 − 41 + 1 = 50
Multiples of 3, 15 to 99	28 + 1 = 29
Odd integers 11 to 49	19 + 1 = 20
Steps from term 1 to term 30	29

IN THE LESSON

- Counting terms causes more lost marks than the formulas themselves, so here is a table to practise.
- The first column is the question, the second is the count.
- All the integers from forty one to ninety: ninety minus forty one plus one, which is fifty.
- Multiples of three from fifteen to ninety nine: we found twenty nine.
- Odd integers from eleven to forty nine: forty nine minus eleven is thirty eight, halved is nineteen, plus one makes twenty.
- And a different question: how many steps does it take to get from the first term to the thirtieth?
- That is twenty nine.
- Spot the pattern: count items, add one.
- Count gaps, no extra one.

21 Geometric sequences: multiply instead

- Geometric: multiply by the same r each time
- 1, 2, 4, 8, 16 has r = 2
- r = 1/2 shrinks; negative r flips the sign
- Multiply n − 1 times to reach term n

IN THE LESSON

- The second kind of sequence multiplies instead of adding.
- Imagine you are offered a penny today, two pennies tomorrow, four the day after, doubling every day.
- The list is one, two, four, eight, sixteen.
- That is a geometric sequence, and the number you multiply by each time is the common ratio, r.
- Here r is two.
- Geometric lists grow, or shrink, much faster than arithmetic ones.
- If the ratio is a half, the terms halve each time.
- If it is negative, the terms flip sign each time.
- The same fence post idea applies.
- To reach the nth term from the first, you multiply n minus one times, so the ratio is used n minus one times.

22 The nth term of a geometric sequence

$a_n = a_1 \times r^{n-1}$

First term (a1)	Where the list starts
Common ratio (r)	Multiplier at each step
Term number n	Which position you want
Term number n (an)	The term you are after

IN THE LESSON

- Here is the formula.
- The nth term equals the first term, times the ratio raised to the power n minus one.
- Again, step by step.
- The first term, a one, is where the list starts.
- The ratio, r, is what you multiply by each time, and you find it by dividing any term by the one before it.
- The exponent, n minus one, is the fence post again: the number of multiplications.
- Check with the pennies.
- The fifth day should be one times two to the power four.
- Two to the power four is sixteen.
- And indeed the list one, two, four, eight, sixteen has sixteen in fifth place.
- The first term is never multiplied by anything on its own.

23 A geometric sequence grows fast

Value of the term

Term 1	**5**
Term 2	**15**
Term 3	**45**
Term 4	**135**

Each bar is three times the one before

IN THE LESSON

- Here is a geometric sequence as bars.
- Read the vertical axis first: it is the value of the term.
- The list starts at five, and the ratio is three.
- The first bar is five.
- The second is fifteen, three times as high.
- The third is forty five, and the fourth is one hundred and thirty five.
- Each bar is three times the one before, so the heights climb steeply.
- An arithmetic list with the same start that added ten each time would reach only thirty five by the fourth term.
- This is why geometric terms get large so quickly, and why you should never try to list them out.
- Use the formula and jump.

24 Example 7: a geometric term

5, 15, 45, … ratio 3. Find the 6th term

Ratio r	$15 \div 5$	**3**
Multiplications	$6 - 1$	**5**
Power of 3	3^5	**243**
6th term	5×243	**1,215**

Trap: $5 \times 3^6 = 3{,}645$ is the 7th term

IN THE LESSON

- Example seven.
- A sequence starts five, fifteen, forty five, and each term is three times the one before.
- What is the sixth term?
- The choices are three hundred and sixty five, seven hundred and twenty nine, twelve hundred and fifteen, thirty six hundred and forty five, and forty three hundred and seventy four.
- Step one, the ratio is fifteen divided by five, which is three.
- Step two, the steps: six minus one is five.
- Step three, three to the power five.
- Three, nine, twenty seven, eighty one, two hundred and forty three.
- Step four, multiply by the first term: five times two hundred and forty three is twelve hundred and fifteen.
- The trap is three to the power six, giving the next term by mistake.

25 Example 8: a rule that repeats

a1 = 2, a2 = 5, then each term = previous − the one before

Terms 1 to 6	2, 5, 3, −2, −5, −3	
Cycle length	then 2, 5, … again	**6**
100 ÷ 6	remainder 4	**4**
100th term	same as the 4th	**−2**

Remainders jump you ahead; a full cycle here sums to 0

IN THE LESSON

- Example eight is a recursive rule, one that builds each term from the ones before.
- The first term is two, the second is five, and every later term is the previous term minus the one before that.
- What is the one hundredth term?
- Do not use a formula.
- Write terms out and watch for a cycle.
- Two, five, then five minus two is three.
- Then three minus five is minus two.
- Then minus two minus three is minus five.
- Then minus five minus minus two is minus three.
- Then minus three minus minus five is two, and then five.
- It repeats every six terms.
- One hundred leaves a remainder of four when divided by six, so the one hundredth term matches the fourth term, minus two.

26 Example 9: a sum that collapses

Sum of 1/(n(n+1)) for n = 1 to 19

Split each term	1/(n(n+1)) = 1/n − 1/(n+1)	
Write it out	(1−½) + (½−⅓) + …	
Middle pieces	cancel in pairs	
Left over	1 − 1/20	**19/20**

IN THE LESSON

- Example nine, a telescoping sum.
- Find the sum of one over n times n plus one, for n from one to nineteen.
- That is one over one times two, plus one over two times three, and so on up to one over nineteen times twenty.
- Adding these directly is a nightmare.
- But each term can be split: one over n times n plus one equals one over n minus one over n plus one.
- Write it out.
- One minus a half, plus a half minus a third, plus a third minus a quarter.
- Each piece cancels the next one, like a telescope folding up.
- Only the very first piece, one, and the very last, minus one over twenty, survive.
- The answer is nineteen twentieths.

27 Mistakes to avoid with sequences

- Off by one: n − 1 steps, last − first + 1 items
- Add the same, or multiply by the same?
- Sum formula needs the count of terms first
- Recursive rule: write terms, find the cycle

IN THE LESSON

- Let us collect the traps for sequences.
- The big one is the off by one error: n minus one steps to reach term n, and last minus first plus one items in a run.
- Second, mixing up the two kinds.
- If the list adds the same amount, it is arithmetic and you use the first formula.
- If it multiplies by the same amount, use the ratio.
- Check by testing two gaps.
- Third, forgetting that the sum formula needs the number of terms, so count first.
- Fourth, a recursive rule: write out several terms and look for a repeating cycle before reaching for any formula.
- Finally, read the question twice.
- Is it asking for the term, the sum, or how many terms?

28 Recap: the whole lesson on one page

- Estimate: round, track direction, compare
- Arithmetic: $a_1 + (n - 1)d$; sum = n × avg of ends
- Geometric: $a_1 \times r^{n-1}$
- Count with n − 1 gaps and +1 items

IN THE LESSON

- Let us finish with a recap.
- Estimation: scan the choices, round to easy numbers, track the direction of each rounding, and compare.
- Box a long sum between the count times the smallest and the count times the largest.
- Count decimal places.
- Never round away the very thing the question asks about.
- When choices are close, use the last digit or odd and even.
- Sequences: in an arithmetic one, the nth term is the first term plus n minus one differences.
- The sum is the count times the average of the ends.
- In a geometric one, the nth term is the first term times the ratio to the power n minus one.
- For recursive rules, write terms and spot the cycle.

29 Try three quick questions

1. Closest to 0.0398 × 5,010?
   Answer: Round to 0.04 × 5,000 = 200.
2. How many integers from 15 to 74 inclusive?
   Answer: 74 − 15 + 1 = 60.
3. Sum of the first 12 terms of 4, 7, 10, …?
   Answer: Last = 4 + 11×3 = 37; sum = 12 × 41 ÷ 2 = 246.

IN THE LESSON

- Now try three quick questions yourself.
- Pause after each one.
- Question one.
- Which is closest to nought point nought three nine eight times five thousand and ten?
- Round to nought point nought four times five thousand, which is two hundred.
- The answer is about two hundred.
- Question two.
- How many integers are there from fifteen to seventy four, inclusive?
- Seventy four minus fifteen plus one is sixty.
- Question three.
- What is the sum of the first twelve terms of four, seven, ten and so on?
- The twelfth term is four plus eleven threes, thirty seven.
- The sum is twelve times forty one, halved, which is two hundred and forty six.