Defined functions and the coordinate plane

Quantitative Reasoning

1 A function is a little machine

- f(x) is a machine: number in, number out
- x is a blank where the input goes
- Rule: f(x) = x² − 2x
- The question prints the rule: just follow it

IN THE LESSON

- Picture a vending machine.
- You press a button, the machine follows its fixed instructions, and out drops one snack.
- A function is exactly that.
- You feed in a number, the rule inside does its work, and one number comes out.
- We write the machine as f of x.
- The letter f is just the name of the machine.
- The letter x is a placeholder, a blank where the input goes.
- If the rule says x squared minus two x, then every time you see an x, you drop your input into that spot.
- On the GMAT exam the question prints the rule for you.
- Your whole job is to copy it faithfully and carry it out.
- Nobody expects you to invent anything.

2 Evaluating: replace every x

f(input) = the rule, with (input) in place of every x

Function name (f)	Which machine to use
Input (x)	The number you drop in
Output f(x)	The value that comes out

IN THE LESSON

- Here is the one rule for evaluating.
- To find f of a number, rewrite the rule and replace every x with that number, wrapped in brackets.
- The three pieces are these.
- The name f tells you which machine you are using, because a question may give you two, f and g.
- The input is the number that goes in the brackets.
- And the output is what comes out, the value of f at that input.
- Why brackets?
- Because they protect signs and powers.
- If the input is minus three, then minus three squared must be written with brackets, and it becomes plus nine.
- Without brackets, you might square only the three and get minus nine.
- Brackets first, always.

3 Problem 1: a plain evaluation

f(x) = x² − 2x. Find f(−3).

Copy the rule		**x² − 2x**
Brackets in	replace each x	**(−3)² − 2(−3)**
Clear the signs	square first	**9 + 6**
Value		**15**

IN THE LESSON

- Problem one, the plain kind.
- If f of x equals x squared minus two x, what is f of minus three?
- The five choices might run from minus three up to twenty one, and the trap answers sit there to catch sign slips.
- Start by copying the rule.
- Replace each x with minus three, in brackets.
- That gives bracket minus three squared, minus two times bracket minus three.
- Now clear the signs.
- Minus three squared is plus nine, because a negative times a negative is positive.
- Minus two times minus three is plus six.
- Nine plus six is fifteen.
- Notice that without brackets you would square only the three, get minus nine, and land on a wrong choice.
- The answer is fifteen.

4 Functions inside functions

- f(g(3)): the inside machine g runs first
- Its output becomes the input for f
- Work inside out, one step at a time
- Order matters: f(g(3)) is not g(f(3))

IN THE LESSON

- Sometimes the question gives two machines and chains them.
- Think of a conveyor belt at a factory.
- One machine paints a part, the next machine wraps it.
- You cannot wrap before you paint.
- In the same way, f of g of three means: put three into g first, and then feed whatever comes out into f.
- So you always work from the inside out.
- The notation reads right to left, which feels backwards, because the machine nearest the number runs first.
- A very common slip is to run them in the wrong order.
- Another is to forget that the output of the first machine is just a number, and the second machine treats it like any other input.

5 Problem 2: a composite

$f(x) = x^2 - 3x$, $g(x) = 2 - x$. Find $f(g(3))$.

Inside first: g(3)	$2 - 3$	**-1**
Feed into f	$f(-1)$	**$(-1)^2 - 3(-1)$**
Clear the signs		**$1 + 3$**
f(g(3))		**4**

IN THE LESSON

- Problem two, the typical kind.
- Let f of x equal x squared minus three x, and g of x equal two minus x.
- What is f of g of three?
- Inside first.
- g of three is two minus three, which is minus one.
- Now that minus one becomes the input for f.
- Bracket minus one squared is one.
- Minus three times bracket minus one is plus three.
- One plus three is four.
- So f of g of three is four.
- Notice how neat it stays when you write every intermediate result on its own line.
- The rows on the board are your scratch paper.
- Never try to hold two substitutions in your head at once.
- On a timed exam that is where the slips come from.

6 Problem 2 twist: swap the order

Same machines, same input 3, opposite order

Chain	Result
f(g(3)): g first, then f	4
g(f(3)): f first, then g	2
Same result?	No

IN THE LESSON

- Now the trap version of the same problem.
- Swap the order and ask for g of f of three.
- Let us see whether the answer is the same.
- Run f first this time.
- f of three is nine minus nine, which is zero.
- Now feed zero into g.
- Two minus zero is two.
- So g of f of three is two, but f of g of three was four.
- Same machines, same starting number, different answers.
- That is the point of the table.
- The order of a composition is part of the question, so read the brackets slowly and underline which one is on the inside.
- If a choice list contains both four and two, the writer is waiting for you to run the machines in the wrong order.

7 Made-up symbols are functions too

- a ★ b = 2a + b is a machine with two slots
- 3 ★ 5: a = 3, b = 5, so 2(3) + 5 = 11
- 5 ★ 3: a = 5, b = 3, so 2(5) + 3 = 13
- New symbols need not be commutative

IN THE LESSON

- Now the second notation.
- The question may invent a symbol, a star or a circle, and define it in one line.
- For example, a star b equals two a plus b.
- It looks strange, but it is the same vending machine, now with two slots.
- Slot one is called a and slot two is called b.
- To work out three star five, copy the rule, put three where a was, put five where b was.
- Two times three plus five is eleven.
- Now swap the inputs.
- Five star three is two times five plus three, which is thirteen.
- Eleven is not thirteen.
- So a new operation cannot be assumed to be commutative, which just means that the order of the inputs may change the answer.

8 Problem 3: a symbol with brackets

a ○ b = ab − a. Find (2 ○ 3) ○ 4.

Inner bracket: 2 ○ 3	2·3 − 2	**4**
Now: 4 ○ 4	4·4 − 4	**12**
Other grouping: 3 ○ 4	3·4 − 3	**9**
Then 2 ○ 9	2·9 − 2	**16**

IN THE LESSON

- Problem three, the trap kind.
- The operation a circle b is defined as a times b, minus a.
- What is the value of bracket two circle three bracket circle four?
- Brackets mean the innermost grouping goes first.
- Two circle three is two times three, minus two, which is four.
- Now use that four as the first input, and the four from the question as the second.
- Four circle four is four times four, minus four, which is twelve.
- A student who suspects that the grouping does not matter might try two circle bracket three circle four bracket instead.
- Three circle four is twelve minus three, which is nine.
- Then two circle nine is eighteen minus two, which is sixteen.
- Twelve against sixteen.
- The grouping matters, so follow the brackets exactly as printed.

9 Problem 4: a rule that builds on itself

a(1) = 3 and a(n+1) = 2·a(n) − 1. Find a(5).

a(2)	2·3 − 1	**5**
a(3)	2·5 − 1	**9**
a(4)	2·9 − 1	**17**
a(5)	2·17 − 1	**33**

IN THE LESSON

- Another style is the repeating rule, also called a recursive rule.
- It says where to start, and how to get each value from the one before.
- Suppose a of one is three, and a of n plus one equals two times a of n, minus one.
- Find a of five.
- You have no shortcut, so build the values one step at a time.
- The first value is three.
- The second is two times three minus one, which is five.
- The third is two times five minus one, which is nine.
- The fourth is seventeen.
- The fifth is two times seventeen minus one, which is thirty three.
- Because it is only five steps, building the list is faster and safer than hunting for a clever formula.
- Write each value down and check the last one.

10 Problem 5: run the machine backwards

f(x) = 3x − 4 and f(k) = 11. Find k.

Put k in the rule	f(k) = 3k − 4	**3k − 4**
Set equal to 11		**3k − 4 = 11**
Add 4, divide by 3	3k = 15	**k = 5**
Check	3·5 − 4	**11**

IN THE LESSON

- Sometimes the output is given and you must find the input.
- If f of x equals three x minus four, and f of k equals eleven, what is k?
- Do not guess.
- Replace x with k in the rule, so f of k is three k minus four.
- Set that equal to eleven.
- Three k minus four equals eleven.
- Add four to both sides, and three k equals fifteen.
- Divide by three, and k is five.
- Then check.
- Three times five is fifteen, minus four is eleven.
- Yes.
- On the exam you can also test the answer choices, plugging each one into the rule, but with a rule this simple the algebra is faster.
- Plugging in is the safer tool when the rule is messy, such as a square.

11 Problem 6: a new input that is an expression

f(x) = x² − 2x. Find f(x + 1), then test x = 3.

Brackets in	replace x by (x+1)	**(x+1)² − 2(x+1)**
Expand	x²+2x+1 − 2x − 2	**x² − 1**
Test x = 3: f(4)	16 − 8	**8**
Test: 3² − 1		**8**

IN THE LESSON

- A harder twist is when the input is not a number but an expression.
- If f of x equals x squared minus two x, what is f of x plus one?
- The same method applies.
- Wherever the rule has an x, drop in bracket x plus one.
- So we get bracket x plus one squared, minus two times bracket x plus one.
- Expand the square: x squared plus two x plus one.
- Expand the second part: minus two x minus two.
- Add them.
- The two x terms cancel, and one minus two leaves minus one.
- So f of x plus one equals x squared minus one.
- A quick check is to try x equals three.
- f of four is sixteen minus eight, which is eight.
- And three squared minus one is eight too.
- A numerical check like this takes ten seconds.

12 The method for every rule question

Functions

Copy the rule → Brackets in for every x → Inside out, line by line

Made-up symbols

Copy the definition → Replace a and b → Innermost group first

IN THE LESSON

- Let us pull the method into one line you can run under time pressure.
- Step one, copy the rule exactly, with its name and its blanks.
- Step two, replace each blank with the given value, inside brackets.
- Step three, if there are brackets or machines inside machines, work from the inside out, and write each result on its own line.
- Step four, clear signs and powers, one operation at a time.
- The bottom row shows the same method for a made up symbol.
- Copy the definition, replace a and b, and do the innermost grouping first.
- The fastest sound path is rarely a trick.
- It is neat writing.
- Most wrong answers on these questions come from skipped lines, not from hard arithmetic.

13 Mistakes to avoid with rules

- Negative input: put it in brackets, $(-3)^2$
- f(g(3)) means g first, then f
- New symbols: not commutative, not associative
- Replace every x, not just the first
- Recursive rule: build every term, never guess

IN THE LESSON

- Here are the traps in this first half.
- First, squaring a negative without brackets.
- Minus three squared written without brackets reads as minus nine, but the input minus three squared is plus nine.
- Second, reversing the order of a composition.
- F of g of three means g runs first.
- Third, assuming that a made up operation behaves like addition.
- It might not be commutative, so swapping the inputs can change the answer, and it might not be associative, so moving the brackets can change it too.
- Fourth, replacing only some of the x terms, which happens when the rule is long.
- Replace every one.
- Finally, rushing a recursive rule by guessing the pattern after two terms.
- Build all the terms the question needs.

14 Now the grid: points and the plane

- Every point is an ordered pair (x, y)
- x first: across. y second: up or down
- (3, 5) and (5, 3) are different points
- A line = all the points that fit one equation

IN THE LESSON

- Now the second skill.
- Think of a street map where every address is two numbers: how far to walk along the road, then how far to walk up.
- The coordinate plane is that map.
- A flat sheet with two number lines crossing at a point called the origin.
- The horizontal line is the x axis, the vertical line is the y axis.
- Every point is an ordered pair, written x comma y, and the order matters.
- Three comma five means go across three, then up five.
- Five comma three is a different spot.
- Negative numbers are to the left of the origin, or below it.
- A straight line is just a long family of points that obey one rule, and the rule is written as an equation.

15 A line drawn: y = 2x + 1

Chart: y vs x

- y-intercept 1
- run 1
- rise 2
- y = 2x + 1

IN THE LESSON

- Here is a line drawn, with x along the bottom and y up the side.
- The line is y equals two x plus one.
- Look where it starts on the left.
- When x is zero, y is one, so it crosses the y axis at height one.
- That crossing is called the y-intercept.
- Now walk along the line.
- From the point where x is one and y is three, step across one unit, which is the run.
- To stay on the line, you must then go up two units, which is the rise.
- Every time you step one to the right, the line climbs by two.
- That steepness, rise over run, is the slope, and here it is two.
- A bigger slope would be a steeper climb.

16 The slope formula

m = rise ÷ run = $(y_2 - y_1) \div (x_2 - x_1)$

Slope (m)	Steepness of the line
Rise ($y_2 - y_1$)	Change in height
Run ($x_2 - x_1$)	Change across

IN THE LESSON

- Here is the formula for the slope between any two points.
- The slope, written m, equals the rise divided by the run.
- The rise is the change in height: the second y minus the first y.
- The run is the change across: the second x minus the first x.
- The golden rule is consistency.
- Whichever point you call first, use it first on top and on the bottom.
- If you take the second y minus the first y on top, you must take the second x minus the first x underneath.
- Mixing the order flips the sign, and the sign is exactly what a trap answer is built on.
- Positive slope climbs to the right, negative slope falls, zero is flat.

17 Problem 7: slope from two points

Line through (−2, 5) and (4, −1). Find the slope.

Rise: $y_2 - y_1$	−1 − 5	**−6**
Run: $x_2 - x_1$	4 − (−2)	**6**
Slope: rise ÷ run	−6 ÷ 6	**−1**
Meaning	falls as x grows	**m = −1**

IN THE LESSON

- Problem seven, a plain one with negatives.
- A line passes through the points minus two comma five and four comma minus one.
- What is its slope?
- Name the points.
- The first point has x equal to minus two and y equal to five.
- The second has x equal to four and y equal to minus one.
- The rise is the second y minus the first y: minus one minus five, which is minus six.
- The run is the second x minus the first x: four minus minus two, which is four plus two, six.
- So the slope is minus six over six, which is minus one.
- A negative slope means the line falls as you move right.
- Subtracting a negative is the usual slip, so write the minus signs out in full.

18 The line: y = mx + b

$y = mx + b$

Height (y)	Height of a point
Slope (m)	Slope: change in y per 1 of x
Across (x)	Across position of a point
Y-intercept (b)	Where it crosses the y axis

IN THE LESSON

- The most useful way to write a line is y equals m x plus b.
- The piece m is the slope we just met.
- The piece b is the y-intercept, the height at which the line crosses the vertical axis, which is where x is zero.
- And x and y are the coordinates of any point on the line.
- So the equation is a rule.
- Give me an x, I multiply by the slope, add b, and out comes the matching y.
- A point is on the line exactly when its coordinates make the equation true.
- To test a point, substitute and see whether both sides agree.
- To read the slope from an equation, the equation must first be solved for y, so that y stands alone on the left.

19 Problem 8: build the equation

Through (−2, 5) and (4, −1), slope −1. Find y = mx + b.

Put in m		**y = −x + b**
Use (−2, 5)	5 = −(−2) + b	**5 = 2 + b**
Solve for b		**b = 3**
Check (4, −1)	−4 + 3	**−1**

IN THE LESSON

- Problem eight.
- Find the equation of the line through minus two comma five and four comma minus one.
- We already know the slope is minus one.
- So y equals minus x plus b.
- We need b.
- Take either point, say minus two comma five, and substitute.
- Five equals minus times minus two, plus b.
- That is five equals two plus b, so b is three.
- The line is y equals minus x plus three.
- Always check with the other point.
- When x is four, y is minus four plus three, which is minus one.
- That matches the second point.
- Notice what we did: slope first, then substitute one point to find b, then confirm with the other point.
- It works on every question of this kind.

20 Problem 9: intercepts from 3x + 2y = 12

3x + 2y = 12. Find both intercepts and the slope.

x-intercept: set y = 0	3x = 12	**(4, 0)**
y-intercept: set x = 0	2y = 12	**(0, 6)**
Solve for y	y = 6 − (3/2)x	**y = −3x/2 + 6**
Slope	from the y form	**−3/2**

IN THE LESSON

- Problem nine, the trap kind.
- A line has the equation three x plus two y equals twelve.
- Find where it crosses each axis, and its slope.
- To find the x-intercept, the point where the line meets the x axis, set y to zero.
- Three x equals twelve, so x is four.
- To find the y-intercept, set x to zero.
- Two y equals twelve, so y is six.
- Now the slope, and here is the trap.
- You cannot read it off the number in front of x.
- First solve for y.
- Two y equals twelve minus three x, so y equals six minus three halves x.
- The slope is minus three halves.
- Check with the intercepts: the line falls six while moving across four, which is minus six over four, the same.

21 That line drawn: 3x + 2y = 12

Chart: y vs x

- y-intercept 6
- x-intercept 4
- slope −3/2
- 3x + 2y = 12

IN THE LESSON

- Let us see that line on the grid.
- It is the line three x plus two y equals twelve.
- Mark the two intercepts first.
- The line crosses the y axis at height six, up near the top left.
- It crosses the x axis at four, out on the right.
- Join them with a straight line.
- Which way does it lean?
- Down and to the right.
- So the slope is negative, as we found.
- And the numbers agree with the picture: going from the top intercept to the right one, you drop six while you move four across.
- Six over four is three halves.
- A quick sketch like this is an excellent check.
- If your slope came out positive, the picture would tell you at a glance that something went wrong.

22 Parallel and perpendicular lines

Partners of a line, by slope

Line's slope	Perpendicular slope
2	−1/2
−3/4	4/3
1/5	−5
Parallel: always the same	same slope

IN THE LESSON

- Two lines can be related in two ways.
- Parallel lines never meet, like railway tracks, so they lean the same amount.
- Equal slopes.
- Perpendicular lines meet at a right angle, like the corner of a page.
- Their slopes are negative reciprocals.
- To build that, flip the fraction upside down, and change the sign.
- A line with slope two has a perpendicular partner with slope minus one half.
- Slope minus three quarters pairs with plus four thirds.
- The table shows these.
- A horizontal line, slope zero, is perpendicular to a vertical line, which has no slope at all, so the rule does not apply to that pair.
- The trap is to flip without changing the sign, or change the sign without flipping.

23 Problem 10: perpendicular and intercept

k: y = 2x + 1. j ⊥ k through (4, 1). Where does j cross the x axis?

Slope of j	−1 ÷ 2	**−1/2**
Find b with (4, 1)	1 = −2 + b	**b = 3**
Line j		**y = −x/2 + 3**
Set y = 0	x/2 = 3	**x = 6**

IN THE LESSON

- Problem ten, the typical kind.
- Line k has the equation y equals two x plus one.
- Line j is perpendicular to k and passes through the point four comma one.
- Where does j cross the x axis?
- Line k has slope two, so j has slope minus one half.
- Write y equals minus a half x plus b.
- Substitute four comma one: one equals minus two plus b, so b is three.
- Line j is y equals minus a half x plus three.
- For the x-intercept set y to zero: zero equals minus a half x plus three, so a half x equals three, and x is six.
- The answer is six.
- Notice the question asked for something three steps away from the slope.
- Keep a note of what is being asked, so you stop at the right place.

24 Problem 11: points on one line

(1, 2), (3, 8), (k, 20) lie on one line. Find k.

Slope of first two	(8 − 2) ÷ (3 − 1)	**3**
Same slope to third	18 ÷ (k − 1) = 3	**k − 1 = 6**
Solve		**k = 7**
Check on y = 3x − 1	3·7 − 1	**20**

IN THE LESSON

- Problem eleven is about collinear points, which means points that lie on one straight line.
- The points are one comma two, three comma eight, and k comma twenty.
- They are on one line.
- Find k.
- The fast sound path is slopes.
- A line has one slope everywhere, so the slope between the first two points must equal the slope between the first and third.
- First two: eight minus two is six, three minus one is two, so the slope is three.
- Now the third: twenty minus two is eighteen, over k minus one.
- Set eighteen over k minus one equal to three.
- Then k minus one is six, and k is seven.
- Check against the line y equals three x minus one.
- At x equals seven, y is twenty.
- Correct.

25 Problem 12: a line in words

Call-out fee $40 plus $25 an hour. Cost y for x hours.

Fixed fee = y-intercept		**b = 40**
Rate per hour = slope		**m = 25**
Equation		**y = 25x + 40**
Three hours	25·3 + 40	**$115**

IN THE LESSON

- The exam often describes a line in a story.
- A plumber charges a fixed call out fee of forty dollars, plus twenty five dollars for each hour of work.
- Write the charge y for x hours, and find the charge for three hours.
- The fixed fee does not change with the hours, so it is the starting height, the y-intercept: forty.
- The twenty five dollars per hour is the rate of change, so it is the slope.
- So y equals twenty five x plus forty.
- For three hours, twenty five times three is seventy five, plus forty is one hundred fifteen dollars.
- Whenever a story gives a fixed amount and a rate, expect a line, with the fixed amount as the intercept and the rate as the slope.

26 Mistakes to avoid with lines

- Same order of subtraction on top and bottom
- Perpendicular: flip the fraction AND change the sign
- Solve for y before reading the slope
- x-intercept: set y = 0. y-intercept: set x = 0
- Collinear: keep each pair of points straight

IN THE LESSON

- Here are the traps for lines.
- First, subtracting coordinates in an inconsistent order.
- If y two comes first on top, x two must come first underneath.
- Second, flipping the slope for a perpendicular line, but forgetting to change the sign, or changing the sign but forgetting to flip.
- Negative reciprocal means both.
- Third, reading the slope from three x plus two y equals twelve as three.
- Solve for y first.
- Fourth, forgetting that the intercepts come from setting the other variable to zero.
- Fifth, in a collinear question, comparing slopes from different starting points without keeping the pairs straight.
- A quick sketch is a check, not proof, so never trust a picture you did not scale.

27 Recap: the whole lesson

- Rules: copy, brackets in, inside out, line by line
- Slope = rise ÷ run, same order top and bottom
- y = mx + b; intercepts: set the other variable to 0
- Parallel: equal slopes. Perpendicular: negative reciprocal

IN THE LESSON

- Let us bring it together.
- For rules you are given: copy the rule, put the input in brackets, work from the inside out, and write every line.
- A made up symbol is only a function with its own name, and it need not be commutative or associative.
- For recursive rules, build the values one step at a time.
- For lines: a point is an ordered pair, slope is rise over run with consistent subtraction, and y equals m x plus b ties them together.
- Intercepts come from setting the other variable to zero.
- Parallel lines share a slope.
- Perpendicular lines have slopes that are negative reciprocals.
- Collinear points share one slope.
- And in words, a fixed amount is the intercept and a rate is the slope.

28 Try three quick questions

1. f(x) = x² − 4x. Find f(−1).
   Answer: (−1)² − 4(−1) = 1 + 4 = 5.
2. a ★ b = 2a − b. Find 5 ★ 3 and 3 ★ 5.
   Answer: 10 − 3 = 7, but 6 − 5 = 1: order matters.
3. Line through (0, 1) and (4, 9). Perpendicular slope?
   Answer: Slope 8 ÷ 4 = 2, so perpendicular is −1/2.

IN THE LESSON

- Now try three quick questions yourself.
- Pause after each one.
- Question one.
- If f of x equals x squared minus four x, what is f of minus one?
- Put minus one in brackets: one plus four, which is five.
- Question two.
- The operation a star b is defined as two a minus b.
- What are five star three and three star five?
- Five star three is ten minus three, which is seven.
- Three star five is six minus five, which is one.
- So the order matters.
- Question three.
- A line passes through zero comma one and four comma nine.
- What slope does a line perpendicular to it have?
- The slope is eight over four, which is two, so the perpendicular slope is minus one half.
- Well done.