CHARTA LEARNING · LESSON NOTES

Quadratics and inequalities

The GMAT exam: Quantitative Reasoning

1 A story: the garden with two answers

- Width x, length x + 3, area 40
- x(x + 3) = 40, so $x^2 + 3x = 40$
- x = 5 works: 5 × 8 = 40
- x = −8 works in the algebra too

IN THE LESSON

- Imagine a rectangular garden.
- It is three metres longer than it is wide, and its area is forty metres squared.
- How wide is it?
- Call the width x.
- The length is x plus three, and width times length is forty.
- So x times x plus three equals forty.
- Multiply out and you get x squared plus three x equals forty.
- That x squared is what makes it a quadratic.
- Now the surprise.
- Equations with an x squared in them usually have two answers, not one.
- Here, x equals five works, because five times eight is forty.
- But x equals minus eight also makes the algebra true.
- The garden only allows the first.
- The exam loves this tension.

2 What a quadratic is

- Highest power of x is two
- Tidy form: $ax^2 + bx + c = 0$
- The answers are called roots: up to two
- Zero times anything is zero, and only that

IN THE LESSON

- A quadratic equation is one where the highest power of x is two.
- Its tidy form puts everything on one side with zero on the other.
- Something times x squared, plus something times x, plus a number, equals zero.
- We call the three pieces the leading coefficient, the middle coefficient and the constant.
- The solutions are called roots, and a quadratic has up to two of them.
- Why does zero matter?
- Because zero is special in multiplication.
- If you multiply two numbers and the answer is zero, then one of them must have been zero.
- Nothing else works.
- That single fact is the engine behind almost every quadratic on the exam, so keep it in your pocket.

3 The zero product rule

$(x - a)(x - b) = 0 \rightarrow x = a$ or $x = b$

First root (a)	Makes the first bracket zero
Second root (b)	Makes the second bracket zero
Zero product rule	One bracket must be zero

IN THE LESSON

- Here is the rule.
- If two brackets multiplied together equal zero, then one bracket is zero, so x is the first root or the second root.
- Take the pieces one at a time.
- The letter a is the number that makes the first bracket zero.
- For the bracket x minus five, a is five.
- The letter b does the same for the second bracket.
- And the rule itself says that when a product is zero, at least one factor is zero.
- Notice the signs.
- The bracket x minus five gives the root plus five, and the bracket x plus four gives the root minus four.
- The sign in the bracket is always the opposite of the root.
- Many wrong answer choices on the exam are exactly this sign slip.

4 The method in four moves

Make one side zero → Factor it → Each bracket equals zero → Reread the question

IN THE LESSON

- Here is the method, and it never changes.
- Move one: get everything onto one side so the other side is exactly zero.
- People skip this, and it is the most common way to go wrong.
- If you see x squared equals five x plus fourteen, do not try to solve it as it stands.
- Move two: factor.
- That means rewrite the expression as two brackets multiplied together.
- Move three: set each bracket equal to zero and solve each one, which gives the roots.
- Move four: read the question again.
- It may ask for the negative root, the sum of the roots, or the larger root, and it may rule one answer out.
- Check your roots by putting them back if you have time.

5 Example 1: a plain one

$x^2 - 3x = 28$. Which values of x?

Move all to one side	$x^2 - 3x - 28 = 0$	
Find the pair	product −28, sum −3	**−7 and 4**
Factor	$(x - 7)(x + 4) = 0$	
Each bracket zero	$x - 7 = 0$ or $x + 4 = 0$	**x = 7 or −4**
Check: roots add to 3	$7 + (-4) = 3 = -b$	**3**

IN THE LESSON

- Example one, a plain one.
- Which values of x satisfy x squared minus three x equals twenty eight?
- The five choices will include the right pair and also pairs with the signs swapped.
- Step one: move twenty eight across, so x squared minus three x minus twenty eight equals zero.
- Step two: find two numbers that multiply to minus twenty eight and add to minus three.
- Try the pairs of twenty eight: one and twenty eight, two and fourteen, four and seven.
- The pair with a difference of three is four and seven, and to get the sum minus three the bigger one is negative, so minus seven and four.
- Step three: x minus seven times x plus four equals zero.
- So x is seven or minus four.
- Check the sum: seven plus minus four is three, which is the opposite of the minus three in front of x.
- It checks out.

6 Squares: do not forget the second root

- $x^2 = 49$ means $x = 7$ or $x = -7$
- Condition $x < 0$ leaves only $x = -7$
- Sum of all solutions of $x^2 = 49$ is 0
- The root sign alone gives only the positive value

IN THE LESSON

- Some quadratics come with no middle term at all.
- Suppose x squared equals forty nine.
- Then x is seven, and also minus seven, because minus seven times minus seven is plus forty nine.
- Always write both.
- On the exam this is often the whole trap.
- A question says x squared equals forty nine and x is less than zero.
- Now there is one answer, minus seven, and the choice that says seven is a decoy.
- Another version asks for the sum of all the solutions.
- With a pair like seven and minus seven the sum is zero, which many people never think of.
- A related fast fact: the square root symbol means the positive root only, so the root of forty nine is seven, while x squared equals forty nine has two answers.

7 Sum and product of the roots

$ax^2 + bx + c = 0$: sum $= -b/a$, product $= c/a$

Number in front of x^2 (a)	Divide b and c by this
Number in front of x (b)	Roots add to minus this over a
Constant term (c)	Roots multiply to this over a

IN THE LESSON

- Here is a shortcut worth learning.
- For a quadratic written in tidy form, the roots add up to minus b over a, and multiply to c over a.
- Let's meet the pieces.
- The letter a is the number in front of x squared, and often it is just one.
- The letter b is the number in front of x.
- The letter c is the plain number at the end.
- So when a is one, the roots add to minus b and multiply to c.
- Look back at example one: b was minus three and c was minus twenty eight, and indeed the roots seven and minus four added to three and multiplied to minus twenty eight.
- When a question asks only for a sum or a product, you can read the answer straight off without factoring at all.

8 Example 2: a leading number that is not one

$2x^2 + 7x - 15 = 0$. Find the roots

Multiply a by c	$2 \times (-15) = -30$	
Pair: product −30, sum 7	10 and −3	
Split the middle	$2x^2 + 10x - 3x - 15$	
Group	$2x(x + 5) - 3(x + 5) = 0$	
Solve	$(2x - 3)(x + 5) = 0$	**$x = 3/2$ or -5**

IN THE LESSON

- Example two, a typical one.
- Two x squared plus seven x minus fifteen equals zero.
- What is the product of the solutions?
- The fast route first.
- Product equals c over a, which is minus fifteen over two.
- Done in five seconds, with no factoring.
- But you should also know how to factor when the leading number is not one.
- Multiply a by c, two times minus fifteen, to get minus thirty.
- Find two numbers that multiply to minus thirty and add to seven: that is ten and minus three.
- Split the middle term into ten x minus three x, group in pairs, and take out common brackets.
- Two x times x plus five, minus three times x plus five.
- The common bracket is x plus five.
- The roots are three over two and minus five.
- Their product is minus fifteen over two, matching the shortcut.

9 A quadratic drawn: where it crosses zero

Chart: $y = x^2 - 4x - 12$ vs x

- x = −2
- x = 6
- Lowest point, x = 2
- y = 0

IN THE LESSON

- Here is the picture behind the roots.
- Along the bottom is x, and up the side is the value of x squared minus four x minus twelve.
- The curve is a bowl shape, called a parabola.
- The flat grey line is the level where the value is zero.
- The roots are exactly the places where the bowl crosses that line.
- Here that happens at minus two on the left and at six on the right.
- Between the two roots the bowl dips below zero, and outside them it is above zero.
- The lowest point sits halfway between the roots, at x equals two.
- Halfway between minus two and six is two.
- This picture will matter again when we solve inequalities.

10 Example 3: the trap that loses a root

(x − 2)(x + 5) = 4(x − 2). Sum of all solutions?

Tempting: divide by (x − 2)	x + 5 = 4	**x = −1 only**
That lost a root	x = 2 makes both sides 0	
Safe: move everything across	(x−2)(x+5) − 4(x−2) = 0	
Take out the common bracket	(x − 2)(x + 1) = 0	
Both solutions, then add	x = 2 or −1	**sum 1**

IN THE LESSON

- Example three, the trap one.
- What is the sum of all values of x for which x minus two times x plus five equals four times x minus two?
- The tempting move is to divide both sides by x minus two.
- Then x plus five equals four, so x is minus one.
- The sum of the solutions would be minus one, and that is sitting in the choices.
- But look what happened.
- You divided by something that could be zero.
- When x is two, both sides are zero, so x equals two is a genuine solution, and the division threw it away.
- The safe way is to move everything across: x minus two times x plus five, minus four times x minus two, equals zero.
- Take out the common bracket x minus two, leaving x plus five minus four, which is x plus one.
- So x is two or minus one, and the sum is one.

11 Example 4: roots that have to be whole numbers

$x^2 - kx + 20 = 0$, roots positive integers

Roots r and s	r × s = 20, r + s = k	
Pairs with product 20	1, 20 2, 10 4, 5	
Each pair's sum is a k	21, 12, 9	
Add the possible k	21 + 12 + 9	**42**

IN THE LESSON

- Example four, a harder one that rewards the sum and product idea.
- Both roots of x squared minus k x plus twenty are positive whole numbers.
- What is the sum of all possible values of k?
- Call the roots r and s.
- Their product is twenty and their sum is k.
- So we list the ways to write twenty as a product of two positive whole numbers: one and twenty, two and ten, four and five.
- Each unordered pair counts once, because one and twenty is the same quadratic as twenty and one.
- The sums are twenty one, twelve and nine, so those are the three possible values of k.
- Their sum is forty two.
- Two traps here.
- Counting the pairs in both orders would double the answer, and twenty is not a perfect square, so there is no repeated root to add.

12 Mistakes to avoid with quadratics

- Not making one side zero first
- Forgetting the second root of a square
- Sign slip: (x + 4) gives root −4
- Dividing by a bracket with x in it
- Forgetting the condition at the end

IN THE LESSON

- Let's collect the mistakes, because on this topic nearly every wrong choice is built from one of them.
- First, solving before you have zero on one side.
- Second, dropping the negative root of a square, so x squared equals nine becomes x equals three only.
- Third, the sign slip: the bracket x plus four gives the root minus four.
- Fourth, dividing both sides by an expression with x in it, which silently deletes a root.
- Whenever you want to divide by something containing x, move it across and factor instead.
- Fifth, forgetting to apply the condition at the end, such as x being negative or being an integer.
- A quick habit saves marks: put your answers back into the original equation.

13 Part two: inequalities, the speed limit idea

- Equation: one answer. Inequality: a range
- Four signs: <, >, ≤, ≥
- A bar under the sign means the end is allowed
- Exam favourite: how many integers fit?

IN THE LESSON

- Now the second skill.
- An equation says two things are equal.
- An inequality says one is bigger than the other.
- Think of a speed limit of ninety kilometres an hour.
- You are fine at eighty, fine at ninety, fine at sixty.
- The rule does not name one speed, it names a whole range.
- That is what an inequality's answer looks like: all the values that make it true.
- We use four signs.
- Less than, greater than, and the two with a bar underneath, which mean less than or equal to, and greater than or equal to.
- The bar means the end value itself is allowed.
- That tiny detail decides how many integers fit in a range, and the exam asks that all the time.

14 The one new rule: negatives flip the line

- −3
- −2
- 2
- 3

IN THE LESSON

- You solve an inequality almost exactly like an equation.
- You may add or subtract the same thing on both sides, and multiply or divide by a positive number, and nothing changes.
- There is one new rule.
- Multiply or divide by a negative number and the sign must flip.
- Here is why.
- On the number line, smaller numbers sit to the left.
- Two is less than three.
- Now multiply both by minus one.
- Two becomes minus two and three becomes minus three.
- But minus two sits to the right of minus three.
- The order has been mirrored, so less than turns into greater than.
- Whenever you divide by a negative, flip the sign.
- Or avoid the flip entirely by moving the x term to the side where it stays positive.

15 Example 5: the flip, and a way to avoid it

9 − 4x ≤ 25. Find all x

Subtract 9	−4x ≤ 16	
Divide by −4 and flip	x ≥ −4	**sign turns**
Other route: keep x positive	9 − 25 ≤ 4x	
Divide by 4: no flip	−16 ≤ 4x, so −4 ≤ x	
Test x = 0 and x = −5	9 ≤ 25 true; 29 ≤ 25 false	**x ≥ −4**

IN THE LESSON

- Example five.
- Solve nine minus four x is less than or equal to twenty five.
- Subtract nine from both sides to get minus four x is less than or equal to sixteen.
- Now divide by minus four.
- Because it is a negative divisor, the sign flips, and we get x is greater than or equal to minus four.
- Here is the second route, which avoids the flip.
- Instead, add four x to both sides and subtract twenty five.
- That gives nine minus twenty five is less than or equal to four x, so minus sixteen is at most four x, and dividing by positive four gives minus four is at most x.
- The same answer.
- Then test one number: x equals zero gives nine, which is at most twenty five, true.
- And x equals minus five gives twenty nine, which is too big, false.

16 Absolute value is distance

$$|x - a| < r \Leftrightarrow a - r < x < a + r$$

Centre (a)	The number x is measured from
Radius (r)	The largest distance allowed
Distance $\lvert x - a \rvert$	How far x is from a

IN THE LESSON

- Absolute value sounds technical but it just means distance from zero.
- The absolute value of x minus a is the distance between x and a.
- So the statement absolute value of x minus a is less than r says: x is within r of a.
- Let's name the pieces.
- The letter a is the centre, the number we measure from.
- The letter r is the radius, the largest distance allowed.
- And the distance itself is the absolute value of x minus a.
- Within r of a means x lies between a minus r and a plus r.
- The other way round, if the distance is greater than r, x is far away, so x is below a minus r or above a plus r.
- Notice the words: between for less than, and two separate pieces joined by or for greater than.

17 Distance drawn: within five of three

Chart: Distance from 3 vs x

- x = −2
- x = 8
- Within 5 of 3
- Limit: 5

IN THE LESSON

- Let's see it.
- Along the bottom is x, and up the side is the distance of x from three.
- The distance is zero at three itself, and it grows in both directions, which draws a V.
- Now we ask: where is the distance at most five?
- Draw a flat line at height five.
- Wherever the V sits below that line, the distance is at most five.
- The V meets the line at minus two on the left and at eight on the right.
- So the answer is every x from minus two to eight.
- Those end points are included, because the distance there is exactly five.
- Where the V rises above the line, x is more than five away, which is the other kind of inequality.

18 Example 6: how many integers?

$|x - 3| \le 5$. How many integers x?

Rewrite as a distance	$-5 \le x - 3 \le 5$	
Add 3 to every part	$-2 \le x \le 8$	
Count: last − first + 1	$8 - (-2) + 1$	**11**
Check both ends	$\lvert -2-3 \rvert = 5$ and $\lvert 8-3 \rvert = 5$	**both in**

IN THE LESSON

- Example six, a plain one.
- How many integers x satisfy the absolute value of x minus three being at most five?
- Read it as a distance: x is within five of three.
- So minus five is at most x minus three, which is at most five.
- Add three to every part, the left, the middle and the right.
- That gives minus two is at most x, which is at most eight.
- Now count.
- The integers run from minus two up to eight.
- The counting rule is the last minus the first, plus one.
- Eight minus minus two is ten, plus one is eleven.
- The classic wrong answer is ten, from forgetting the plus one.
- Check the ends: minus two is five away from three, and eight is five away.
- Both are allowed, because of the bar under the sign.

19 Example 7: a typical one with a coefficient

$|2x - 5| < 9$. How many integers x?

Squeeze the inside	$-9 < 2x - 5 < 9$	
Add 5 to every part	$-4 < 2x < 14$	
Divide every part by 2	$-2 < x < 7$	
Count: both ends excluded	$-1, 0, 1, \ldots, 6$	**8 integers**

IN THE LESSON

- Example seven, a typical one.
- How many integers x satisfy the absolute value of two x minus five being less than nine?
- This time the thing inside is two x minus five, not just x minus something.
- No problem.
- The distance rule works for any inside.
- Squeeze it: minus nine is less than two x minus five, which is less than nine.
- Add five to every part: minus four is less than two x, which is less than fourteen.
- Divide every part by two, which is positive so there is no flip: minus two is less than x, which is less than seven.
- Now the signs are strict, so both ends are out.
- The integers are minus one, zero, one, and on up to six.
- Count them: six minus minus one is seven, plus one is eight.
- Strict ends are why the answer is not nine.

20 Four forms of the distance rule

Centre 4, radius 3

Form	Meaning
$\lvert x - 4\rvert < 3$	$1 < x < 7$
$\lvert x - 4\rvert \le 3$	$1 \le x \le 7$
$\lvert x - 4\rvert > 3$	$x < 1$ or $x > 7$
$\lvert x - 4\rvert \ge 3$	$x \le 1$ or $x \ge 7$

IN THE LESSON

- Here is a small table to remember.
- We take the centre as four and the radius as three, so the numbers are easy.
- Absolute value of x minus four less than three means x between one and seven, both ends out.
- With less than or equal to, the ends come in.
- Now the flip side.
- Greater than three means x is more than three away from four, so x is below one or above seven, with the two ends out.
- And greater than or equal to three brings the ends in.
- Notice the pattern.
- Less than means one connected stretch, joined by the word and.
- Greater than means two pieces pointing away from each other, joined by the word or.
- Mixing up and with or is a very common slip, so say the words aloud as you write.

21 Example 8: the far-away version

$\lvert x + 1\rvert > 4$, integers from -10 to 10

Split into two	$x + 1 > 4$ or $x + 1 < -4$	
Solve each	$x > 3$ or $x < -5$	
Integers 4 to 10	$10 - 4 + 1$	**7**
Integers −10 to −6	$-6 - (-10) + 1$	**5**
Add the two groups	$7 + 5$	**12**

IN THE LESSON

- Example eight.
- For how many integers x from minus ten to ten, inclusive, is the absolute value of x plus one greater than four?
- Greater than means far away, so split it into two pieces joined by or.
- Either x plus one is greater than four, or x plus one is less than minus four.
- Solve each: x is greater than three, or x is less than minus five.
- Now count inside the allowed range.
- On the right, the integers from four up to ten: ten minus four is six, plus one is seven.
- On the left, from minus ten up to minus six: minus six minus minus ten is four, plus one is five.
- Seven plus five is twelve.
- A quick check: the integers that fail run from minus five to three, which is nine, and twenty one in all minus nine is twelve.

22 Ranges: why you cannot subtract end to end

$2 \le x \le 6$ and $1 \le y \le 4$. Range of $x - y$?

Smallest x − y	smallest x, biggest y	**2 − 4 = −2**
Largest x − y	biggest x, smallest y	**6 − 1 = 5**
So x − y lies from	−2 up to 5	**−2 to 5**
Wrong: pair the ends	2 − 1 and 6 − 4	**1 to 2**

IN THE LESSON

- One more favourite.
- Suppose x is between two and six, and y is between one and four.
- What range can x minus y take?
- The trap is to subtract end by end: two minus one and six minus four, giving one to two.
- That is wrong.
- To make x minus y as small as possible, take the smallest x and subtract the biggest y: two minus four is minus two.
- To make it as large as possible, take the biggest x and subtract the smallest y: six minus one is five.
- So x minus y runs from minus two to five.
- The reason is that subtracting y flips its role, a cousin of the negative flip we met.
- A big y makes the difference small.
- Always pair the opposite ends when you subtract.

23 Back to quadratics: a quadratic inequality

$x^2 - x - 12 < 0$. How many integers x?

Factor	$(x - 4)(x + 3)$	
Roots split the line	$x = 4$ and $x = -3$	
Test x = 0 between them	$(-4)(3) = -12$, negative	**in**
So the solution is	$-3 < x < 4$	
Integers, ends excluded	−2, −1, 0, 1, 2, 3	**6 integers**

IN THE LESSON

- Now the two skills meet.
- Solve x squared minus x minus twelve is less than zero.
- First treat it like an equation: x squared minus x minus twelve factors as x minus four times x plus three.
- The roots are four and minus three.
- Those two points split the number line into three zones.
- The bowl shape we drew earlier tells us what happens: between the roots the bowl dips below zero, outside it is above.
- We want less than zero, so the answer is between the roots.
- Test a number to be sure: x equals zero gives minus twelve, which is negative, so zero is in.
- The answer is minus three less than x less than four, strict.
- The integers are minus two up to three, which is six of them.

24 Traps in inequalities

- Forgetting to flip with a negative
- Strict versus inclusive ends
- Integers: last − first + 1
- And versus or; ranges subtract crosswise

IN THE LESSON

- Let's gather the traps.
- First and biggest, forgetting to flip the sign when you multiply or divide by a negative.
- Second, confusing strict and inclusive ends: with a plain less than the end is out, with the bar it is in, and that changes the integer count by one at each end.
- Third, counting integers wrongly: it is last minus first plus one, not just last minus first.
- Fourth, writing and when the answer needs or, or the other way round.
- Fifth, subtracting ranges end by end.
- And a general rule for questions that ask what must be true: test the boundaries and test a negative number and zero.
- If a statement fails for even one allowed value, it is not always true.

25 The decision method for inequalities

No absolute value

Collect x on one side → Divide, flip if negative → Test a number

With absolute value

Less than: squeeze →
Greater than: split in two →
Count with the ends

IN THE LESSON

- Here is the whole method on one board.
- If there is no absolute value, collect x on one side, divide to finish, and flip the sign only if you divided by a negative.
- Then test one number from inside and one from outside.
- If there is an absolute value, ask whether it is less than or greater than.
- Less than means squeeze: the inside is between minus r and r, so the answer is one connected stretch.
- Greater than means split: the inside is above r or below minus r, two pieces.
- Finally, if the question asks how many integers, write out the first and last that qualify, watch the ends, and count last minus first plus one.

26 Recap

- Quadratic: zero, factor, brackets, condition
- Sum of roots −b/a, product c/a
- Inequality: flip for a negative
- |x − a| < r: between. |x − a| > r: outside

IN THE LESSON

- Let's recap both skills.
- For quadratics: get zero on one side, factor, set each bracket to zero, and then apply any condition in the question.
- The sign in the bracket is opposite to the root.
- For squares, always write the two roots.
- The roots add to minus b over a and multiply to c over a, which answers sum and product questions at once.
- Never divide by an expression containing x, because you will lose a root.
- For inequalities: solve like equations but flip the sign when you multiply or divide by a negative.
- Absolute value is distance: less than means between, greater than means outside, joined by or.
- Count integers as last minus first plus one, and check each end.

27 Try three quick questions

1. Solve $x^2 - 2x - 15 = 0$.
   Answer: $x = 5$ or $x = -3$.
2. Solve $3 - 2x < 11$.
   Answer: $-2x < 8$, divide by -2 and flip: $x > -4$.
3. How many integers satisfy $|x + 2| \le 4$?
   Answer: $-6 \le x \le 2$: that is 9 integers.

IN THE LESSON

- Now try three quick questions yourself.
- Pause after each one.
- Question one.
- Solve x squared minus two x minus fifteen equals zero.
- We need two numbers with product minus fifteen and sum minus two.
- Those are minus five and plus three.
- So x minus five times x plus three equals zero, and x is five or minus three.
- Question two.
- Solve three minus two x is less than eleven.
- Subtract three to get minus two x less than eight, then divide by minus two and flip the sign, so x is greater than minus four.
- Question three.
- How many integers satisfy the absolute value of x plus two being at most four?
- X plus two sits between minus four and four, so x runs from minus six to two, and the count is nine.