CHARTA LEARNING · LESSON NOTES

Rate problems and work problems

Quantitative Reasoning

1 A story: the road trip

- Speed 60 mph: 60 miles for each hour
- A rate = an amount for each one of something
- Three questions: how far, how long, how fast
- One sum, with one piece hidden

IN THE LESSON

- Picture a long drive.
- You glance at the dashboard and it says sixty miles per hour.
- What does that tell you?
- It says that for every hour on the road, you cover sixty miles.
- That is a rate: an amount of one thing for each one of another.
- Speed is distance for each hour.
- Rates are everywhere.
- Pages read per minute, litres filled per second, jobs finished per day.
- Once you can see a rate, you can answer three questions.
- How far will I go?
- How long will it take?
- How fast must I be?
- All three are the same sum, written three ways, and the exam only changes which piece it hides.

2 The one formula

D = R × T

Distance, D	How far: miles, km
Rate (speed), R	How far in ONE hour
Time, T	How long: hours

IN THE LESSON

- Here is the formula that runs every trip.
- Distance equals rate times time.
- Take each term.
- Distance is how far you travel, in miles or kilometres.
- Rate is the speed, how far you go in one unit of time, such as miles per hour.
- Time is how long you travel.
- If the speed is fifty and you drive three hours, the distance is one hundred and fifty.
- Turn it around when something else is hidden.
- Rate is distance divided by time.
- Time is distance divided by rate.
- Many students keep a small triangle in their head, with distance on top, and rate and time below.
- Cover the one you want, and read off the rest.

3 Units come first

- Time must match the rate: hours for mph
- 15 min = 1/4 hour, 30 min = 1/2 hour
- 20 min = 1/3 hour, 40 min = 2/3 hour
- 45 minutes is 0.75 hour, NOT 0.45

IN THE LESSON

- Now the trap that catches the most marks: units.
- A rate says miles per hour, so the time must be in hours.
- If a question gives minutes, convert before you multiply.
- Forty-five minutes is three quarters of an hour, which is zero point seven five.
- It is not zero point four five.
- Many people write the minutes after a decimal point, and that is wrong, because an hour has sixty minutes, not one hundred.
- Here is a quick table.
- Fifteen minutes is one quarter.
- Thirty minutes is one half.
- Twenty minutes is one third.
- Forty minutes is two thirds.
- Learn those four fractions and you will rarely need to divide by sixty on the exam.

4 Example 1: a plain trip

A train: 54 mph for 3 hours 20 minutes. How far?

Time in hours	3 h 20 min = 3 1/3 h	**3 1/3**
Rate	given	**54 mph**
Whole hours	54 × 3	**162**
Distance	162 + 54 ÷ 3 = 162 + 18	**180 miles**

IN THE LESSON

- Example one, the plain kind.
- A train travels at fifty-four miles per hour for three hours and twenty minutes.
- How far does it go?
- First the units.
- Twenty minutes is one third of an hour, so the time is three and a third hours.
- Now multiply.
- Fifty-four times three is one hundred and sixty-two.
- Fifty-four times one third is eighteen.
- Add them: one hundred and eighty miles.
- Notice the speed was chosen so that it divides neatly by three.
- On the exam, that is a hint.
- When the numbers are friendly, use fractions of an hour rather than decimals, and the work stays small.
- The trap answer here is multiplying by three point two, which gives a wrong figure.

5 Example 2: finding the speed

20 km in 40 minutes. Speed in km per hour?

Time in hours	40 min = 2/3 hour	**2/3**
Rate = D ÷ T	20 ÷ 2/3 = 20 × 3/2	**30**
Check: one full hour	20 + half of 20	**30 km**
Trap: per minute	20 ÷ 40	**0.5**

IN THE LESSON

- Example two, now with the speed hidden.
- A cyclist rides twenty kilometres in forty minutes.
- What is the cyclist's speed in kilometres per hour?
- Rate is distance divided by time, and the time must be in hours.
- Forty minutes is two thirds of an hour.
- Twenty divided by two thirds is twenty times three halves, which is thirty.
- So the cyclist rides at thirty kilometres per hour.
- There is a quicker way to see it.
- Forty minutes is two thirds of an hour, so in a full hour they would ride half as far again, twenty plus ten, which is thirty.
- The trap answer is zero point five, which is kilometres per minute.
- The question asked per hour, so check the unit in the answer choices.

6 Average speed is not the average of speeds

- Fast out, slow back: you spend longer slow
- So the slow speed counts for more
- Use totals: all the distance, all the time
- Never average the two speeds

IN THE LESSON

- Now a famous trap.
- Imagine a drive to the coast and back.
- On the way out the road is clear, and you go fast.
- On the way back it is jammed, and you go slow.
- Your average speed for the whole journey is not halfway between the two.
- Why not?
- Because you spend longer on the slow part.
- The slow speed lasts for more hours, so it counts for more.
- The only honest way is to use totals.
- Add up all the distance.
- Add up all the time.
- Divide.
- Whenever a question gives two speeds over the same distance, your first thought should be: do not average the speeds, work out each time.

7 Average speed

Average speed = Total distance ÷ Total time

Average speed	For the WHOLE trip
Distance, D	Add every leg
Time, T	Add every leg, rests too

IN THE LESSON

- Here is the rule.
- Average speed equals total distance divided by total time.
- Total distance is every mile of the whole journey, outward leg and return leg added together.
- Total time is every hour spent, including the time at each speed, and including any rest stop if the question says the time covers it.
- Divide the first by the second.
- That is the whole method, and it works for a trip with two parts, three parts or ten.
- Notice what it does not say.
- It does not say add the speeds and halve them.
- That works only in the special case where you spend equal time at each speed, which we will test on the next slide.

8 Example 3: out and back

120 km out at 40 km/h, 120 km back at 60 km/h

Time out	120 ÷ 40	**3 hours**
Time back	120 ÷ 60	**2 hours**
Total distance, total time	240 km, 5 hours	**240 and 5**
Average speed	240 ÷ 5	**48 km/h**

IN THE LESSON

- Example three.
- A van drives one hundred and twenty kilometres to a depot at forty kilometres per hour, then returns by the same road at sixty.
- What is its average speed for the round trip?
- Work leg by leg.
- Going out takes one hundred and twenty divided by forty, which is three hours.
- Coming back takes one hundred and twenty divided by sixty, which is two hours.
- The total distance is two hundred and forty kilometres, and the total time is five hours.
- Two hundred and forty over five is forty-eight.
- So the answer is forty-eight, and the tempting answer, fifty, is exactly the trap.
- Forty-eight is lower, because the van spent three hours at the slow speed and only two at the fast.

9 When can you just average?

Same two speeds, 40 and 60 km/h, two situations

Situation	Average speed
1 hour at 40, then 1 hour at 60	50 km/h
120 km at 40, then 120 km at 60	48 km/h
Plain average of 40 and 60	50 km/h
Rule: equal TIMES may be averaged	equal DISTANCES not

IN THE LESSON

- Why is fifty wrong in that example, yet right sometimes?
- Look at the table.
- If the van spends equal time at each speed, say one hour at forty and one hour at sixty, it covers forty plus sixty, one hundred kilometres, in two hours.
- That is fifty.
- The plain average works.
- But if it covers equal distances, one hundred and twenty kilometres each way, the slow part takes longer, and the answer is forty-eight.
- The question wording tells you which case you are in.
- Equal times: you may average.
- Equal distances, or any two legs of the same length: you may not.
- When in doubt, use totals, and you can never go wrong.

10 Two travellers: the closing speed

Time to meet = Gap ÷ Closing speed

Gap between them	Distance between them now
Closing speed	Same way: subtract. Towards: add
Time, T	Hours until they meet

IN THE LESSON

- Next, two travellers at once.
- Imagine two friends walking on a long straight path.
- If they walk the same way and the back one is faster, the gap between them shrinks.
- By how much each hour?
- By the difference of their speeds.
- If they walk towards each other, the gap shrinks by the sum of their speeds, because each one eats into the gap.
- That shrinking rate is called the closing speed.
- Time to meet equals the gap divided by the closing speed.
- Same direction: subtract the speeds.
- Opposite directions towards each other: add.
- If they walk apart, the gap grows by the same sum, so the sum is the separation speed.

11 Picture: catching up

Chart: Miles from start vs Hours after 8:00

- Head start: 40 miles
- Level at 11:00

IN THE LESSON

- Here is a head start, drawn.
- The bottom axis is time, in hours after eight in the morning.
- The side axis is miles from the starting point.
- The first line is a truck that leaves at eight and climbs steadily at forty miles per hour.
- The second line is a faster truck, sixty miles per hour, which only leaves at nine.
- So its line starts one quarter of the way along, and it is steeper.
- Look at the first mark.
- When the second truck sets off, the first is already forty miles ahead.
- That vertical gap is the head start.
- The second line climbs faster, and the lines cross at the second mark, at eleven o'clock.
- There the trucks are level.

12 Example 4: the chase

Truck A: 8:00 at 40 mph. Truck B: 9:00 at 60 mph.

Head start at 9:00	40 mph × 1 hour	**40 miles**
Closing speed	60 − 40	**20 mph**
Time to catch up	40 ÷ 20, timed from 9:00	**2 hours**
Where, and check	B: 60 × 2; A: 40 × 3	**120 miles**

IN THE LESSON

- Example four.
- A truck leaves a depot at eight in the morning at forty miles per hour.
- A second truck leaves the same depot, on the same road, at nine, at sixty miles per hour.
- When does the second truck catch the first, and how far from the depot?
- Step one, the head start.
- By nine, the first truck has gone forty miles.
- Step two, the closing speed: same direction, so subtract, sixty minus forty is twenty.
- Step three, the time: forty divided by twenty is two hours.
- But from when?
- From nine, when the second truck left, so the catch happens at eleven.
- The distance is sixty times two, one hundred and twenty miles.
- The wrong-start trap would say three hours.

13 Example 5: towards each other

330 km apart. Train A leaves 9:00 at 60; train B leaves 10:00 at 90.

Gap at 10:00	330 − 60 × 1	**270 km**
Closing speed	60 + 90, towards	**150 km/h**
Time to meet	270 ÷ 150	**1.8 hours**
From city one	60 × (1 + 1.8)	**168 km**

IN THE LESSON

- Example five, with a late start and opposite directions.
- Two cities are three hundred and thirty kilometres apart.
- A train leaves the first city at nine at sixty kilometres per hour, heading to the second.
- At ten, a second train leaves the second city, heading back at ninety.
- How far from the first city do they meet?
- By ten the first train has gone sixty kilometres, so the gap left is two hundred and seventy.
- They now head towards each other, so the closing speed is sixty plus ninety, one hundred and fifty.
- Two hundred and seventy divided by one hundred and fifty is one point eight hours.
- The first train has then travelled two point eight hours, so one hundred and sixty-eight kilometres.
- As a check, the second train covers ninety times one point eight, one hundred and sixty-two, and the two add up to three hundred and thirty.

14 Example 6: hitting a target average

120 km at an average of 48 km/h; first 60 km at 40

Total time allowed	120 ÷ 48	**2.5 hours**
Time already used	60 ÷ 40	**1.5 hours**
Time left, distance left	2.5 − 1.5; 120 − 60	**1 h, 60 km**
Speed needed	60 ÷ 1	**60 km/h**

IN THE LESSON

- Example six, a favourite on the exam: hit a target average.
- A courier must cover one hundred and twenty kilometres and wants an average speed of forty-eight kilometres per hour.
- The first sixty kilometres go at forty.
- How fast must the courier drive the rest?
- Do not average.
- Find the total time allowed: one hundred and twenty divided by forty-eight is two and a half hours.
- Subtract the time already used: sixty divided by forty is one and a half hours.
- One hour is left.
- The remaining distance is sixty kilometres.
- Sixty kilometres in one hour needs sixty kilometres per hour.
- The trap answer is fifty-six, which comes from wrongly making forty and the answer average to forty-eight.

15 The method for any rate problem

Method

Units: all in hours →
Sketch who, when, which way →
Totals, or gap and closing speed →
Reread: what, and from when?

IN THE LESSON

- Let us put the method on one picture.
- Along the top, the order I would follow on every trip question.
- First, units: bring every time into hours, and every distance into the same unit as the speed.
- Second, sketch: a line, the start, who leaves when, and which direction.
- Third, totals or gap: for one traveller in stages, list each leg's time.
- For two travellers, find the gap at the moment the second one starts.
- Then divide by the closing speed.
- Last, reread the question.
- Is it asking for a time, a distance, a speed?
- And timed from when?
- The last check catches most of the careless errors.

16 Mistakes in rate problems

- 1. Averaging speeds instead of using totals
- 2. Minutes written as decimals of an hour
- 3. Timing from the wrong starting moment
- 4. Adding when you should subtract, or reverse

IN THE LESSON

- Here are the four traps in this half.
- One: averaging the speeds.
- Two speeds over the same distance never average to the middle.
- Use total distance over total time.
- Two: mixing units.
- Minutes are not decimals of an hour, and a speed in kilometres per hour needs a time in hours.
- Three: timing from the wrong start.
- If the second traveller leaves later, the catch-up time runs from their departure, not from the first start.
- Four: adding speeds when they chase each other, or subtracting when they head towards each other.
- Chase: subtract.
- Towards: add.
- The exam often layers two of these in one question, so read slowly and go in order.

17 A story: painting the fence

- You: 10 hours. Friend: 15 hours.
- Together must beat both, so not 12.5
- Rate = jobs per hour = 1 ÷ time
- Working together: the RATES add

IN THE LESSON

- Now the second half, jobs.
- Imagine you can paint a fence in ten hours, and your friend can paint the same fence in fifteen.
- How long does it take if you both paint together?
- Do not say twelve and a half, the average.
- You are getting help, so it must be faster than either of you alone.
- The key is to think in jobs per hour, not hours per job.
- If you finish a whole fence in ten hours, then in one hour you finish one tenth of the fence.
- That is your rate.
- Your friend's rate is one fifteenth.
- Now the powerful fact: when people work together at steady rates, their rates add.

18 Combine the rates

Rate together = 1 ÷ A + 1 ÷ B

Rate of A, 1 ÷ A	Fraction of the job per hour
Rate of B, 1 ÷ B	Fraction of the job per hour
Rate together	Add them. Time = 1 ÷ this

IN THE LESSON

- Here is the rule written out.
- If worker A takes a certain number of hours alone, A's rate is one divided by that number: the fraction of the job done in one hour.
- Same for B.
- Working together, their rate is the sum of the two rates.
- The time together is then one divided by that combined rate.
- Notice we add rates, never times.
- A tank with a drain is the same idea.
- Something working against the job, like a drain emptying while a tap fills, has a negative rate, so you subtract it.
- For two workers there is also a useful shortcut.
- Time together equals the product of the two times, divided by their sum.

19 Example 7: the fence

You: 10 hours alone. Friend: 15 hours alone.

Your rate, friend's rate	1/10 and 1/15	**3/30, 2/30**
Rate together	3/30 + 2/30	**5/30 = 1/6**
Time together	1 ÷ 1/6	**6 hours**
Shortcut check	10 × 15 ÷ (10 + 15)	**6 hours**

IN THE LESSON

- Example seven, the fence.
- You take ten hours, your friend takes fifteen.
- Your rate is one tenth of the job each hour.
- Your friend's is one fifteenth.
- Add them.
- The common denominator is thirty.
- One tenth is three thirtieths, one fifteenth is two thirtieths, and together that is five thirtieths, which simplifies to one sixth of the job per hour.
- If one sixth is done each hour, the whole job takes six hours.
- The shortcut agrees: ten times fifteen is one hundred and fifty, divided by ten plus fifteen, twenty-five, which is six.
- Check that six is sensible: it is less than ten, so help has helped.
- The trap answers are twelve and a half, the average, and twenty-five, the sum.

20 Pick a job size

- Job size = a number every time divides into
- 10 and 15: take 30 panels
- You: 3 an hour. Friend: 2 an hour
- Together 5 an hour: 30 ÷ 5 = 6 hours

IN THE LESSON

- Fractions can feel heavy without a calculator, so here is a shortcut that the exam loves.
- Pick a size for the job that makes every rate a whole number.
- Choose the smallest number that every time divides into, the least common multiple.
- For ten and fifteen, that is thirty.
- Say the fence is thirty panels.
- You paint thirty divided by ten, three panels an hour.
- Your friend paints thirty divided by fifteen, two panels an hour.
- Together, five panels an hour, and thirty panels take six hours.
- Same answer, no fractions.
- This is allowed because the question never tells you the real size of the job.
- Whatever size you choose, the time comes out the same.
- Just stay with your chosen size all the way through.

21 Picture: the rates add

Panels per hour

You	**3**
Friend	**2**
Together	**5**

Fence counted as 30 panels

IN THE LESSON

- Here are those rates as bars.
- Each bar is how many panels per hour, when the fence is counted as thirty panels.
- The first bar is you, at three panels an hour, because thirty panels in ten hours is three.
- The second bar is your friend, at two, because thirty in fifteen is two.
- Now the third bar, together: it is as tall as the first two stacked end to end, five panels an hour.
- That is what it means for rates to add.
- Then the time follows.
- Thirty panels at five an hour is six hours.
- Notice how the shorter job time, ten hours, belongs to the taller bar.
- A faster worker has a bigger rate and a smaller time.

22 Example 8: a tap and a drain

Pipe fills in 5 h; drain empties in 20 h; both open

Tank size	LCM of 5 and 20	**20 units**
Pipe, drain	20 ÷ 5; 20 ÷ 20	**+4, −1 per hour**
Net rate	4 − 1	**3 per hour**
Time to fill	20 ÷ 3	**6 h 40 min**

IN THE LESSON

- Example eight, with something working against the job.
- A tank is filled by a pipe in five hours, but a drain at the bottom empties a full tank in twenty hours.
- With both open, how long to fill an empty tank?
- The least common multiple of five and twenty is twenty, so count the tank as twenty units.
- The pipe adds twenty divided by five, four units an hour.
- The drain removes twenty divided by twenty, one unit an hour.
- The net rate is four minus one, three units an hour.
- Twenty units at three an hour is six and two thirds hours, which is six hours and forty minutes.
- The trap is to ignore the drain, and answer five.

23 Example 9: a helper joins late

Anna: 12 h alone. Ben: 6 h alone. Ben joins after 5 hours.

Hall = 12 units	Anna 1 per hour, Ben 2	**1 and 2**
Done before Ben joins	1 × 5	**5 units**
Left, rate together	12 − 5; 1 + 2	**7 units, 3/h**
Time still needed	7 ÷ 3	**2 h 20 min**

IN THE LESSON

- Example nine, where the crew changes partway.
- Anna can paint a hall alone in twelve hours.
- Ben can paint it alone in six.
- Anna starts alone, and Ben joins her after five hours.
- How much longer until the hall is done?
- Count the hall as twelve units.
- Anna paints one unit an hour.
- Ben paints two.
- In the first five hours Anna alone does five units, leaving seven.
- Now both are working, at three units an hour.
- Seven divided by three is two and a third hours, which is two hours and twenty minutes.
- That is the answer to how much longer.
- The total time would be seven hours twenty.
- Read the question: it asked how much longer, not the total.

24 Example 10: find one worker's time

Together 4 h. Anna alone 6 h. Carl alone?

Rate together	1 ÷ 4	**1/4**
Anna's rate	1 ÷ 6	**1/6**
Carl's rate	1/4 − 1/6 = 3/12 − 2/12	**1/12**
Carl's time	1 ÷ 1/12	**12 hours**

IN THE LESSON

- Example ten, running the problem backwards, which exam writers like.
- Anna and Carl together finish a job in four hours.
- Anna alone takes six hours.
- How long would Carl take alone?
- Here the combined rate is known, so subtract.
- Together, one quarter of the job per hour.
- Anna alone, one sixth.
- Carl's rate is the difference: one quarter minus one sixth, which is three twelfths minus two twelfths, one twelfth.
- One twelfth of the job per hour means twelve hours for the whole job.
- With a job size of twelve, it is even faster: together, three units an hour; Anna, two; so Carl is one unit an hour, and twelve units take twelve hours.
- The trap answer is two, from subtracting the times.

25 Crews: worker-days

Work = Workers × Days

Work, in worker-days	Total effort of the job
Number of workers	Identical workers
Days taken	Days each one works

IN THE LESSON

- Now crews of identical workers.
- A bigger crew finishes sooner, but the total effort stays the same.
- Measure effort in worker-days: the number of workers times the number of days.
- Work equals workers times days.
- If twelve workers need ten days, that job is one hundred and twenty worker-days.
- More output needs more worker-days, in proportion.
- Double the road, double the worker-days.
- And finishing the same output in fewer days needs more workers: the number of workers and the days are inversely related, with the product fixed.
- So first find the worker-days for each unit of output, then scale to the new output, then divide by the new crew.

26 Example 11: a road crew

12 workers: 3 km in 10 days. 15 workers: 4.5 km?

Worker-days for 3 km	12 × 10	**120**
Worker-days per km	120 ÷ 3	**40**
Worker-days for 4.5 km	40 × 4.5	**180**
Days for 15 workers	180 ÷ 15	**12 days**

IN THE LESSON

- Example eleven.
- A crew of twelve identical workers lays three kilometres of road in ten days.
- How many days would fifteen workers need to lay four and a half kilometres?
- Step one, the worker-days.
- Twelve times ten is one hundred and twenty for three kilometres.
- So each kilometre costs forty worker-days.
- Step two, scale.
- Four and a half kilometres cost forty times four and a half, one hundred and eighty worker-days.
- Step three, divide by the new crew: one hundred and eighty over fifteen is twelve days.
- A sense check: more road, more days, but more workers too, so a little more than ten is sensible.
- The trap is to scale only the road and forget the larger crew.

27 Mistakes in work problems

- 1. Adding or averaging times, not rates
- 2. Forgetting work done before a change
- 3. Total time versus how much longer
- 4. Drain or leak: subtract its rate

IN THE LESSON

- Four traps in work problems.
- First, adding or averaging the times instead of the rates.
- Ten and fifteen hours never combine into twenty-five or twelve and a half.
- Convert to rates, add, then invert.
- Second, forgetting the work already done before the crew changed.
- Always count how much of the job is finished at each stage, subtract it from the whole, and only then divide by the new rate.
- Third, answering the wrong question.
- Total time and the extra time after a change are different numbers, and the exam offers both as choices.
- Fourth, a drain or leak.
- Remember its negative rate, so subtract it.

28 Recap: three rules

D = R × T Avg = total D ÷ total T Together = 1/A + 1/B

Distance, D	Distance = rate × time
Average speed	Total distance ÷ total time
Closing speed	Gap ÷ closing speed
Rate together	Add the RATES, not times
Work, in worker-days	Workers × days

IN THE LESSON

- Let us pull it together.
- For every trip: distance equals rate times time, and put the time in hours.
- For a trip in stages: average speed is total distance over total time, never the plain average of speeds.
- For two travellers: find the gap when the second one starts, then divide by the closing speed, subtract the speeds if they head the same way, and add them if they head towards each other.
- For jobs: rates add, so one over A plus one over B gives the combined rate, and a drain subtracts.
- For crews: work equals workers times days.
- If you can say each piece in plain words, you can solve almost any question of this kind.

29 Try three quick questions

1. A car goes 48 km/h for 45 minutes. How far?
   Answer: 3/4 × 48 = 36 km
2. Cyclists 60 km apart ride towards each other at 12 and 8 km/h. Hours to meet?
   Answer: 60 ÷ (12 + 8) = 3 hours
3. Machine A takes 12 hours, machine B takes 4. Together?
   Answer: 1/12 + 1/4 = 1/3, so 3 hours

IN THE LESSON

- Now try three questions.
- Pause after each.
- Question one.
- A car travels at forty-eight kilometres per hour for forty-five minutes.
- How far does it go?
- Forty-five minutes is three quarters of an hour, and three quarters of forty-eight is thirty-six kilometres.
- Question two.
- Two cyclists are sixty kilometres apart and ride towards each other, at twelve and eight kilometres per hour.
- When do they meet?
- The closing speed is twenty, and sixty divided by twenty is three hours.
- Question three.
- Machine A fills an order in twelve hours and machine B in four.
- How long together?
- A twelfth plus a quarter is a third, so three hours.
- Well done.