Ratios, proportions and percents

Quantitative Reasoning

1 A story: the fruit punch

- Ratio 3 : 5 compares amounts, not totals
- Juice 3 cups for every 5 cups of soda
- Give both the same helper number k
- Juice = 3k, soda = 5k

IN THE LESSON

- Imagine you are making fruit punch for a party.
- The recipe says three cups of orange juice for every five cups of soda.
- That is a ratio.
- A ratio is simply a comparison of amounts, and we say it as three to five.
- Notice what it does not tell you.
- It does not say how much punch you make.
- You could fill a jug or a bath, as long as the juice and soda keep the same balance.
- So here is the big idea for the GMAT exam.
- Never work with the bare ratio.
- Turn it into real amounts using one helper number, which we call k, the size of one part.
- Three to five becomes three k and five k.

2 The ratio rule: parts and one multiplier

Ratio a : b → amounts ak and bk; whole = (a + b)k

First term (a)	Ratio's first number
Second term (b)	Ratio's second number
One part (k)	Size of one part
Whole = (a + b)k	Whole: all the parts

IN THE LESSON

- Here is the rule.
- If two amounts are in the ratio a to b, then the first amount is a times k and the second is b times k.
- Let us take the terms one at a time.
- The letters a and b are the two numbers in the ratio, the first term and the second term.
- The letter k is the size of one part, and it is the same for both amounts.
- That is why the method works.
- The whole is the sum, a plus b, all times k.
- So the punch with three to five has eight parts.
- One piece of information about any single amount, or about the whole, fixes k, and then everything else follows.

3 Example 1: red and blue marbles

Red to blue is 3 : 5, and 48 marbles in all. How many blue?

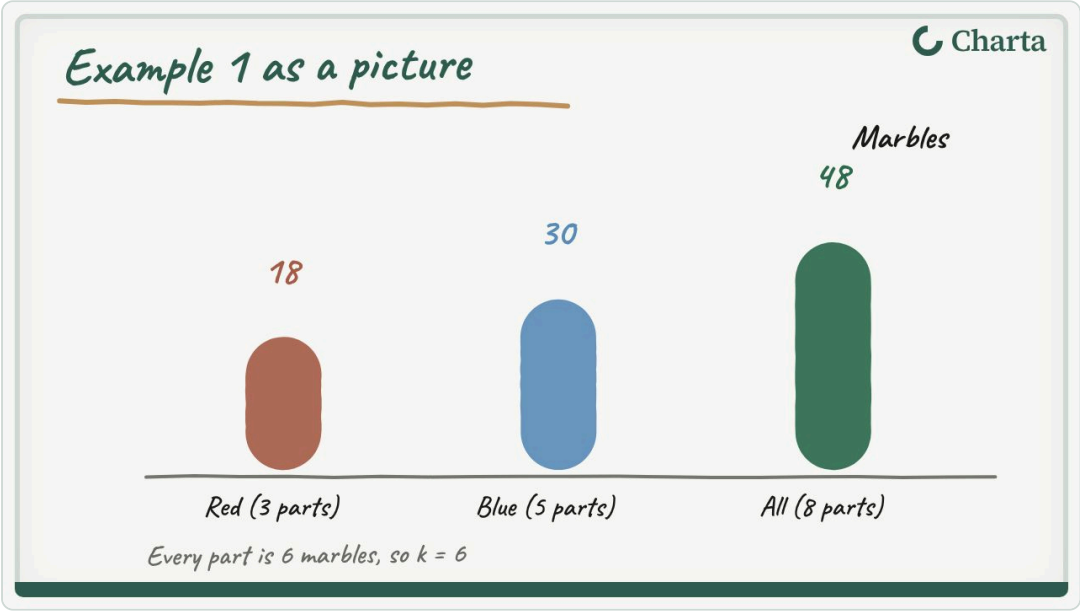
Write the parts	red 3k, blue 5k	
The whole	3k + 5k = 8k = 48	**k = 6**
Blue	5k = 5 × 6	**30**
Check	red 18 + blue 30	**48**

Choice A, 18, is the red count: right number, wrong question

IN THE LESSON

- Example one, a plain one.
- A bag holds red and blue marbles in the ratio three to five.
- There are forty eight marbles in all.
- How many are blue?
- The choices are eighteen, twenty four, twenty eight, thirty and thirty two.
- Step one, write the amounts as three k and five k.
- Step two, the whole is eight k, and that equals forty eight.
- Step three, divide to find k, which is six.
- Step four, blue is five times six, which is thirty.
- As a check, red is three times six, eighteen, and eighteen plus thirty is forty eight.
- The trap is choice A, eighteen.
- That is the red count, a correct number answering the wrong question.

4 Example 1 as a picture



IN THE LESSON

- Here is the same bag as bars.
- Look at the baseline first.
- The height of each bar is a number of marbles.
- The first bar is red, three parts of six marbles each, so eighteen.
- The second is blue, five parts of six, so thirty.
- The tall bar on the right is all eight parts together, forty eight.
- Notice how the blue bar is five thirds the height of the red one.
- That is the ratio, five to three, still sitting there inside the real numbers.
- Whenever the answer must be a whole number of marbles, the total has to be a multiple of eight.
- Forty eight is.
- A total of fifty would have told you straight away that something was wrong.

5 Proportions: two ratios that match

- Proportion: two equal ratios
- Same quantity on top of both fractions
- Map: 2 cm per 5 km, so 4 cm is 10 km
- Scale by the factor, or cross-multiply

IN THE LESSON

- A proportion says that two ratios are equal.
- Think of a photocopier set to enlarge.
- Every length on the original is multiplied by the same factor, so the shape stays the same.
- If a map shows two centimetres for every five kilometres, then four centimetres must be ten kilometres.
- The ratio of map length to real distance has not changed.
- On the GMAT exam, the safest habit is to keep the same quantity on the top of both fractions.
- Put juice over soda on both sides, never juice over soda on one side and soda over juice on the other.
- Mixing up the order is the most common slip.
- Once the two fractions are lined up, you can scale, or cross multiply.

6 The proportion rule: cross multiply

a ÷ b = c ÷ d → a × d = b × c

First amount (a)	Top of the first ratio
Its partner (b)	Bottom of the first ratio
New first amount (c)	Top of the second ratio
New partner (d)	Bottom of the second ratio

IN THE LESSON

- Here is the rule for solving a proportion.
- If a over b equals c over d, then a times d equals b times c.
- We call it cross multiplication.
- Take the terms.
- The letters a and b are the first ratio, with a on top and b below.
- The letters c and d are the second ratio, in the same order, with c on top and d below.
- Multiply across the diagonals and set the two products equal.
- Usually one of the four letters is missing, so you get a one line equation.
- A faster way is to look for the scale factor.
- If the bottom doubled, the top doubled.
- Spotting a friendly factor saves time, with no calculator.

7 Example 2: concentrate and water

Concentrate to water is 3 : 7. Water used: 35 litres.

Same quantity on top	3/7 = x/35	
Cross-multiply	7x = 3 × 35 = 105	
Solve	x = 105 ÷ 7	**15 litres**
Check	15 : 35 = 3 : 7	**matches**

Shortcut: 35 is 7 × 5, so 3 × 5 = 15

IN THE LESSON

- Example two.
- A drink mixes concentrate and water in the ratio three to seven.
- A batch uses thirty five litres of water.
- How much concentrate does it need?
- The choices are twelve, fifteen, eighteen, twenty and twenty five.
- Step one, write concentrate over water as three over seven, and set it equal to x over thirty five.
- Water is on the bottom both times.
- Step two, cross multiply to get seven x equals one hundred and five.
- Step three, divide by seven to get fifteen litres.
- The shortcut is to spot that thirty five is seven times five, so scale the three by five and you get fifteen at once.
- The check is fifteen to thirty five, which reduces to three to seven.

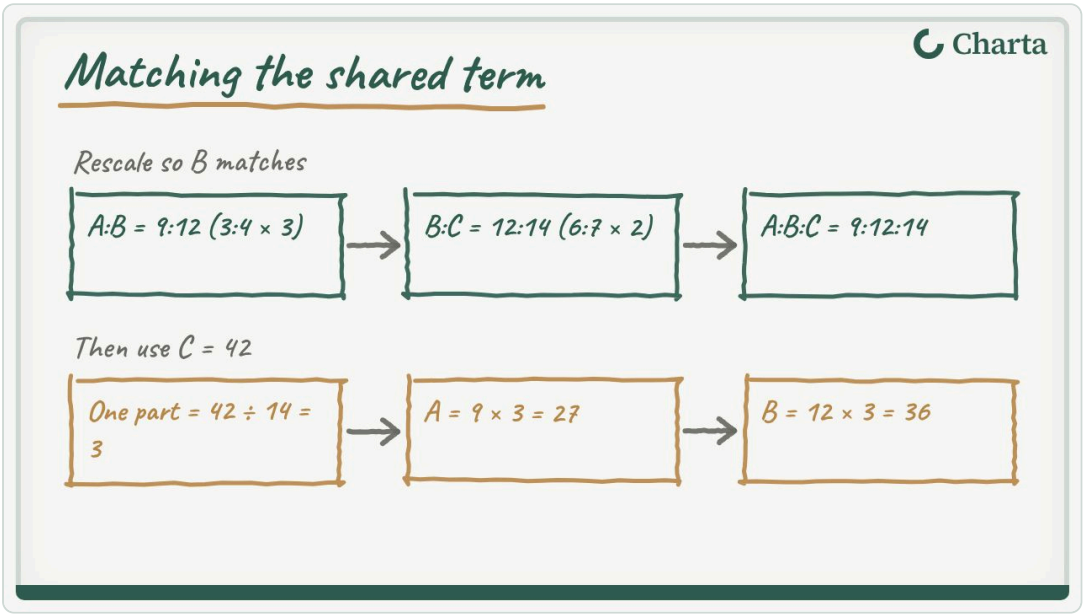
8 Linking two ratios through a shared term

- A : B and B : C share the term B
- B must be the same number in both
- Rescale to a common multiple of B
- Then read A : B : C in one line

IN THE LESSON

- Sometimes you get two ratios that share a member, such as the ratio of A to B, and the ratio of B to C.
- Imagine two friends comparing sweets.
- Anna has two sweets for every three of Ben's.
- Ben has four for every five of Chloe's.
- Ben appears in both, but he counts as three in one comparison and four in the other.
- You cannot read A to C off directly.
- The fix is to rescale so that the shared term is the same in both.
- The smallest number that both three and four go into is twelve, so we stretch each ratio until Ben is twelve.
- Then all three can be written in one chain, and the shared term can be ignored.

9 Matching the shared term



IN THE LESSON

- Follow the boxes.
- In the top row, A to B is three to four, and B to C is six to seven.
- The B's do not match.
- B is four in the first ratio and six in the second.
- The smallest number that four and six both go into is twelve.
- So multiply the first ratio by three, which gives nine to twelve, and the second by two, which gives twelve to fourteen.
- Now B is twelve in both, and the chain reads nine, twelve, fourteen.
- The bottom row puts it to work.
- If C is forty two, then fourteen parts are forty two, so one part is three.
- A is nine parts, twenty seven.
- B is twelve parts, thirty six.
- The trap is reading A to C as three to seven.

10 Example 3: sharing money in a chain

A:B = 2:3, B:C = 4:5, total $70. How much does C get?

Match B at 12	8:12 and 12:15	**A:B:C = 8:12:15**
Parts in all	8 + 12 + 15	**35**
One part	$70 ÷ 35	**$2**
C's share	15 × $2	**$30**

Trap: reading A:C as 2:5 gives the wrong split

IN THE LESSON

- Example three, a typical one.
- Money is shared among A, B and C.
- The ratio of A to B is two to three, the ratio of B to C is four to five, and the total is seventy dollars.
- How much does C get?
- The choices are twenty four, twenty eight, thirty, thirty five and forty two.
- Step one, match B.
- Three and four go into twelve, so A to B becomes eight to twelve, and B to C becomes twelve to fifteen.
- Step two, the chain is eight, twelve, fifteen, thirty five parts.
- Step three, seventy divided by thirty five is two dollars a part.
- C gets fifteen parts, thirty dollars.
- A tempting error is reading A to C as two to five.

11 When amounts change after the ratio is set

- The ratio describes the START amounts
- Start: write them with k
- Changes are fixed numbers: add or subtract
- New ratio gives one equation in k

IN THE LESSON

- Now the harder kind of ratio question, and the one the GMAT exam loves.
- A ratio is given, then something happens.
- People leave, stock is sold, a delivery arrives.
- Think of a school trip.
- At the start the boys and girls are in some ratio.
- Then a few boys go home and a few girls arrive.
- The ratio of the new numbers is different.
- The key idea is that the ratio describes the starting amounts, so the starting amounts are written with k.
- The changes are fixed numbers, so they are added or subtracted.
- Boys become five k minus something, girls become four k plus something.
- The new ratio then gives you one equation, and you can solve it with ease.

12 Example 4: a school trip

Boys : girls = 5 : 4. Then 10 boys leave, 5 girls join; now equal.

At the start	boys 5k, girls 4k	
After the changes	5k − 10 and 4k + 5	
Now equal	5k − 10 = 4k + 5	**k = 15**
Students at the start	9k = 9 × 15	**135**
Check: after	65 boys, 65 girls	**130**

130 is the total AFTER the changes: the tempting wrong choice

IN THE LESSON

- Example four, a trap one.
- On a school trip the ratio of boys to girls is five to four.
- Ten boys go home and five girls join.
- Now there are equal numbers of boys and girls.
- How many students were on the trip at the start?
- Choices include one hundred and thirty, one hundred and thirty five, and one hundred and forty.
- Step one, boys are five k and girls four k.
- Step two, after the changes, boys are five k minus ten, girls are four k plus five.
- Step three, equal means five k minus ten equals four k plus five, so k is fifteen.
- Step four, the start was nine k, one hundred and thirty five.
- The trap is one hundred and thirty, the total after the changes.

13 Example 5: more workers, less time

6 workers take 10 days. How long for 15 workers?

Total work	6 workers × 10 days	**60 worker-days**
Same job, 15 workers	60 ÷ 15	**4 days**
Wrong direction	10 × 15 ÷ 6	**25 days**

More workers should mean fewer days: sense-check the direction

IN THE LESSON

- Example five shows that not every proportion goes the same way.
- Six workers finish a job in ten days, all working at the same pace.
- How long do fifteen workers take?
- Think about the story.
- More workers should mean fewer days, not more.
- So this is an inverse proportion, where the product stays fixed.
- Six workers for ten days is sixty worker days of work.
- That job always needs sixty worker days.
- With fifteen workers, sixty divided by fifteen is four days.
- The trap is to scale in the same direction.
- Ten times fifteen over six is twenty five days, which makes no sense when you add helpers.
- Always ask whether the answer should be bigger or smaller before you finish.

14 Mistakes to avoid with ratios

- 1. Add the same to both terms: ratio changes
- 2. A part is not the whole: 3 : 5 has 8 parts
- 3. More workers, less time: inverse
- 4. Keep the order: 'boys to girls' = boys first

IN THE LESSON

- Four mistakes to avoid.
- One.
- Adding or subtracting the same number to both terms and thinking the ratio stays the same.
- Three to five becomes four to six if you add one to each, and that is a different ratio.
- Ratios survive multiplying and dividing, not adding.
- Two.
- Treating a part as the whole.
- Three to five is eight parts, so the juice is three eighths of the punch, not three fifths.
- Three.
- Scaling the wrong way, more workers, less time.
- Four.
- Losing the order.
- If the question says boys to girls, the first number belongs to the boys.
- Underline the order words as you read, and the slip disappears.

15 Percents: out of one hundred

- Percent = per hundred = a fraction of 100
- 25% means 25 squares of a 100-square bar
- Same number as a fraction or a decimal
- Pick the form that makes the sum easy

IN THE LESSON

- Now percents.
- The word means per hundred, so thirty per cent is thirty out of every hundred.
- Picture a bar of chocolate with a hundred squares.
- Twenty five per cent is twenty five squares, a quarter of the bar.
- Every percent is also a fraction and a decimal, so it can be written three ways, and the trick is to pick the one that makes the sum easiest.
- With no calculator, a handful of friendly fractions will carry you a long way.
- Fifty per cent is a half, twenty five is a quarter, twenty is a fifth, and ten is a tenth.
- Learn these and many questions become mental arithmetic.

16 Friendly percents to remember

Percents you should know as fractions

Percent	Fraction
50%	1/2
25%	1/4
20%	1/5
10%	1/10

IN THE LESSON

- Here is the cheat sheet.
- Look at the left column, the percent, and the right column, the fraction.
- Fifty per cent is one half.
- Twenty five per cent is one quarter.
- Twenty per cent is one fifth.
- Ten per cent is one tenth.
- Why does this matter?
- Suppose you need twenty per cent of sixty five.
- A fifth of sixty five is thirteen, with no long multiplication.
- Or you can find ten per cent by moving the point, then double it.
- Fifteen per cent of eighty is ten per cent, eight, plus half of that, four, so twelve.
- Building a percent from ten per cent and five per cent is a classic, quick method.

17 Every percent change is a multiplier

New = Old × (1 ± r ÷ 100)

Old value	The starting value
Rate of change (r)	Percent rise (+) or fall (−)
Multiplier	1.15 for +15%, 0.70 for −30%
New value	Old value × multiplier

IN THE LESSON

- Here is the idea that unlocks this whole topic.
- Every percent change is a multiplier.
- An increase of r per cent multiplies the old value by one plus r over one hundred.
- A decrease of r per cent multiplies by one minus r over one hundred.
- Let us take the terms.
- The old value is what you start with.
- The letter r is the rate of change, in percent.
- The multiplier is the number you multiply by.
- A fifteen per cent rise multiplies by one point one five.
- A thirty per cent fall multiplies by zero point seven.
- The new value is the old value times the multiplier.
- Seeing percents as multipliers turns tricky word problems into one step of multiplication.

18 Percent change: always over the old value

Percent change = (new − old) ÷ old × 100

New value	Where it ended up
Old value	Where it started: the base
Change = new − old	Positive up, negative down
Percent change	Change as a part of the old

IN THE LESSON

- Now the other direction.
- How do we measure a percent change when we are given two values?
- The percent change equals the change divided by the old value, times one hundred.
- Take the terms.
- The new value and the old value are given.
- The change is the new minus the old, so it is positive for a rise and negative for a fall.
- Then divide by the old value.
- That last step is the whole secret.
- The base for a percent change is always the starting value, never the new one and never the bigger one.
- The first question to ask is always, a percent of what?
- Say the answer to yourself before you calculate.
- The base is the starting value.

19 Example 6: the shirt that went up

Price rises from $80 to $92. What percent increase?

The change	92 − 80	**$12**
The base	the old price	**$80**
Percent change	12 ÷ 80 × 100	**15%**
Mental check	8 (10%) + 4 (5%) = 12	**15%**

Choice A, 12, confuses dollars with percent

IN THE LESSON

- Example six, a plain one.
- A shirt costs eighty dollars.
- After a price rise it costs ninety two.
- What is the percent increase?
- The choices are twelve, thirteen, fourteen, fifteen and sixteen.
- Step one, the change is ninety two minus eighty, which is twelve.
- Step two, the base is the old price, eighty.
- Step three, twelve over eighty is zero point one five, so fifteen per cent.
- There is a no calculator route.
- Ten per cent of eighty is eight, five per cent is four, and eight plus four is twelve, so the rise is ten plus five, fifteen per cent.
- The trap is choice A, twelve, which is the dollar change mistaken for a percent.
- It only works when the base is one hundred.

20 Successive changes: a story

- Up 20%, then down 20%: back to start?
- No: the fall is 20% of a BIGGER price
- Each change works on the latest value
- So successive percents multiply, not add

IN THE LESSON

- Now the heart of the topic, successive changes.
- Suppose a shop raises the price of a game by twenty per cent, then a month later cuts it by twenty per cent.
- Is it back where it began?
- Most people say yes.
- The answer is no.
- Think of it as walking up a hill and down.
- The rise is twenty per cent of the original price.
- But the fall is twenty per cent of the higher price, so it takes off more.
- The second change works on the new value, not on the start.
- This is why successive percents never add.
- They multiply.
- Each change takes the latest value as its base, and the final price ends up a little lower than the start.

21 Up twenty, then down twenty



IN THE LESSON

- Here are the numbers.
- Start with the first bar.
- The value begins at one hundred.
- The second bar is after a twenty per cent rise.
- Twenty per cent of one hundred is twenty, so it climbs to one hundred and twenty.
- The third bar is after a twenty per cent fall.
- The fall is twenty per cent of one hundred and twenty, which is twenty four, not twenty.
- So it ends at ninety six.
- That is four below where we began, a four per cent loss.
- The multiplier method gets the same answer in one line: one point two times zero point eight is zero point nine six.
- And notice the order does not matter.
- Fall first, then rise, and you also end at ninety six.

22 A shortcut for two changes

Net % change = a + b + (a × b) ÷ 100

First change (a)	First change, in percent
Second change (b)	Second change, in percent
Cross term (ab ÷ 100)	The part people forget
Net percent change	Total change, in percent

IN THE LESSON

- When there are exactly two changes, there is a neat shortcut.
- If the two changes are a per cent and b per cent, the net change is a plus b plus a times b over one hundred.
- Take the terms.
- The letters a and b are the first and second changes, with a minus sign for a fall.
- The last piece is the cross term, a times b over one hundred.
- It is the part people forget, the extra caused by the second change working on the changed value.
- Try twenty and minus twenty.
- Twenty minus twenty is zero, and the cross term is minus four, so the net change is minus four.
- Two rises of ten per cent give ten plus ten plus one, twenty one per cent.

23 Example 7: the phone price chain

$200, raised 30%, then cut 10%. Final price?

Multipliers	1.30 and 0.90	
Net factor	1.30 × 0.90	**1.17**
Final price	200 × 1.17	**$234**
Shortcut check	30 − 10 + (30 × −10) ÷ 100	**+17%**

Choice D, 240, adds the percents: wrong

IN THE LESSON

- Example seven, a typical one.
- A phone costs two hundred dollars.
- The price is raised thirty per cent and then cut ten per cent.
- What is the final price?
- The choices are two hundred and twenty, two hundred and thirty, two hundred and thirty four, two hundred and forty and two hundred and sixty.
- Step one, write the multipliers, one point three and zero point nine.
- Step two, multiply them to get one point one seven.
- Step three, two hundred times one point one seven is two hundred and thirty four.
- A rise of seventeen per cent overall.
- The trap is choice D.
- Thirty minus ten is twenty, and twenty per cent of two hundred gives two hundred and forty.
- That is adding successive percents, which is wrong.

24 Example 8: working back to the start

20% off, then 5% tax. Paid $168. Sticker price?

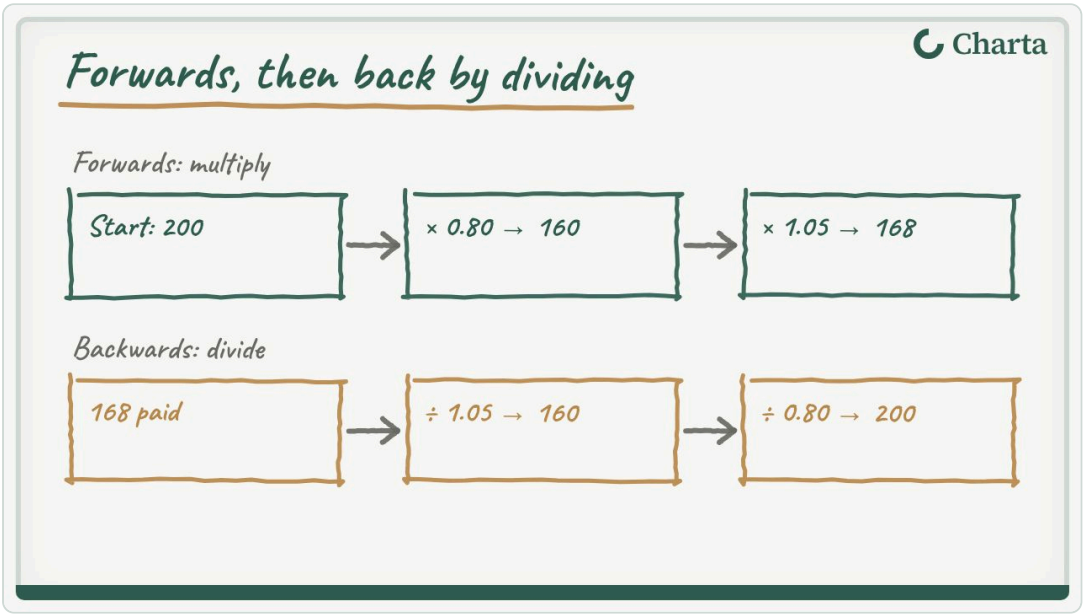
Multipliers	0.80 and 1.05	
Net factor	0.80 × 1.05	**0.84**
Undo: divide	168 ÷ 0.84	**$200**
Check forwards	200 → 160 → 168	**matches**

Net is −16%, not −15%: percents multiply, they do not add

IN THE LESSON

- Example eight, a trap one, because the start is unknown.
- A coat is on sale for twenty per cent off, and then five per cent sales tax is added.
- The customer pays one hundred and sixty eight dollars.
- What was the sticker price before the sale?
- Step one, the multipliers are zero point eight and one point zero five.
- Step two, the net factor is zero point eight four.
- Step three, to undo a multiplier you divide, so one hundred and sixty eight over zero point eight four is two hundred.
- The check is that two hundred becomes one sixty, and one sixty becomes one sixty eight.
- The trap is netting the percents as minus fifteen, which gives an awkward answer, a sure sign the base slipped.

25 Forwards, then back by dividing



IN THE LESSON

- Here is the coat as a picture, so the undo step sticks.
- The top row goes forwards.
- Start at two hundred.
- Multiply by zero point eight for the discount, and you have one hundred and sixty.
- Multiply by one point zero five for the tax, and you pay one hundred and sixty eight.
- The bottom row goes backwards.
- To undo the tax, divide by one point zero five, and you are back to one hundred and sixty.
- To undo the discount, divide by zero point eight, and you are back to two hundred.
- Divide in the reverse order, or all at once by the net factor.
- Never undo a percent by taking the same percent off the answer.
- Adding back five per cent of one sixty eight does not work.

26 Example 9: more than, less than

Mia earns 25% more than Raj. Raj earns what % less than Mia?

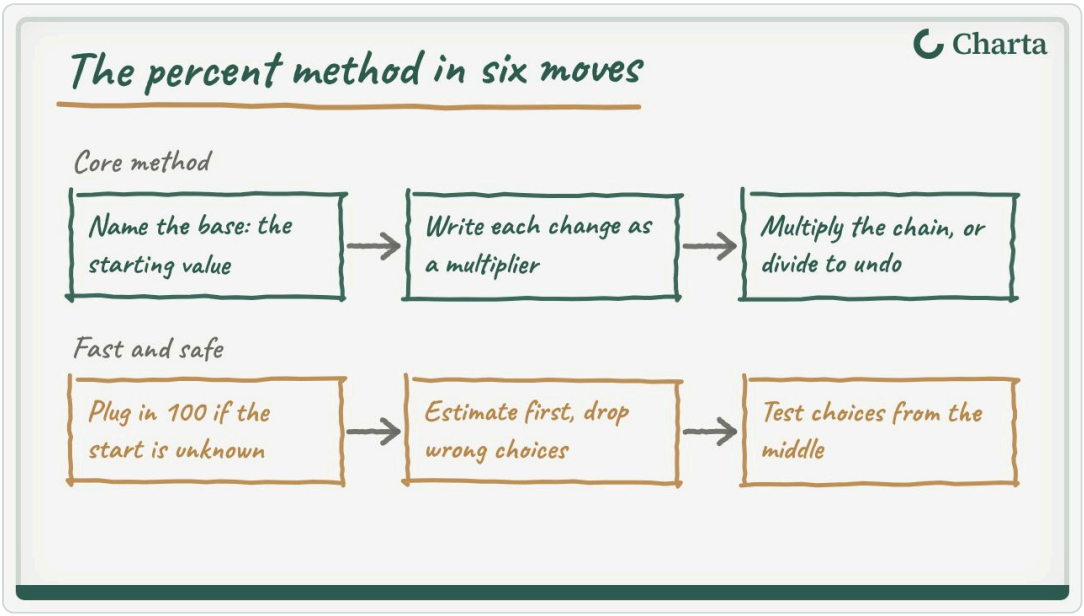
Plug in	Raj 100, so Mia 125	
Gap	125 − 100	**25**
New base is Mia	25 ÷ 125	**20%**
Wrong reflex	reuse the 25	**25%**

'More than' uses Raj as the base; 'less than' uses Mia

IN THE LESSON

- Example nine shows how the base flips.
- Mia earns twenty five per cent more than Raj.
- Raj earns what percent less than Mia?
- The choices include fifteen, twenty, twenty five, and thirty.
- The fastest sound path is to plug in one hundred for the person you start with.
- If Raj earns one hundred, Mia earns one hundred and twenty five.
- Now ask the new question.
- Raj is twenty five below Mia, and the base is now Mia, so twenty five over one hundred and twenty five, which is one fifth, twenty per cent.
- The wording more than has one base, less than has another.
- The tempting wrong answer is twenty five, reusing the first percent on the other base.

27 The percent method in six moves



IN THE LESSON

- Here is the method as a picture, to use on any percent question.
- The top row is the core.
- First, name the base.
- Ask what the percent is a percent of, and it is nearly always the starting value, or whatever the phrase after of, than, or from refers to.
- Second, turn each change into a multiplier.
- Third, multiply the chain, or divide when you are undoing it.
- The bottom row is for speed and safety.
- If the starting value is unknown, plug in one hundred.
- Round and estimate before you calculate, so you can throw away wrong choices.
- And test the choices from the middle, so you waste no time when the sums are messy.
- Together these make a one minute path.

28 Mistakes to avoid with percents

- 1. Successive percents multiply: +30, −10 is +17
- 2. The base is the OLD value, not the new
- 3. To undo a change, divide by the multiplier
- 4. 20% more than B is not B being 20% less

IN THE LESSON

- Four mistakes cost the most marks.
- One.
- Adding successive percents.
- Up thirty then down ten is plus seventeen, not plus twenty.
- Two.
- Using the new value as the base.
- A percent change is always over the old value, the starting one.
- Three.
- Taking a percent off to undo a percent on.
- To undo, divide by the multiplier.
- Four.
- Mixing up percent of with percent more than.
- A is one point two times B means A is twenty per cent more than B, but B is only about sixteen point seven per cent less than A.
- Whenever a sum looks awkward, check the base.
- A messy answer is often a slipped base.

29 Recap: five rules to keep

Ratios and percents in five rules

Ratios	a : b → ak and bk
Proportions	a/b = c/d → ad = bc
Multipliers	Change = multiplier 1 ± r/100
Percent change	(new − old) ÷ old × 100
Chains	Multiply the multipliers

IN THE LESSON

- Let us pull it together.
- For ratios, turn the comparison into parts: a to b becomes a k and b k, and one known amount fixes k.
- For proportions, keep the same quantity on top of both fractions, then cross multiply or scale, and ask whether the answer should get bigger or smaller.
- For percents, every change is a multiplier, one plus or minus r over one hundred.
- Percent change is the change over the old value.
- And for chains, multiply the multipliers, never add the percents, and divide to undo.
- Underneath all five sits one habit: before you calculate, ask what the base is, and what the answer should look like.

30 Try three quick questions

1. x : y = 4 : 7 and x + y = 88. Find y.
   Answer: 11 parts = 88, k = 8, so y = 7 × 8 = 56.
2. Up 50%, then down 50%. Net change?
   Answer: 1.5 × 0.5 = 0.75, so a 25% fall.
3. After a 20% rise the price is $90. Original?
   Answer: 90 ÷ 1.2 = $75.

IN THE LESSON

- Now try three quick questions yourself.
- Pause after each one.
- Question one.
- The ratio of x to y is four to seven, and x plus y is eighty eight.
- What is y?
- Eleven parts make eighty eight, so one part is eight, and y is seven times eight, fifty six.
- Question two.
- A value rises fifty per cent, then falls fifty per cent.
- What is the net change?
- One point five times zero point five is zero point seven five, so it is a twenty five per cent fall.
- Question three.
- After a twenty per cent rise, a price is ninety dollars.
- What was the original price?
- Divide ninety by one point two, and you get seventy five dollars.
- Well done.