Charta | LESSON NOTES

Backsolving: let the choices do the work

Quantitative Reasoning

1 Working backwards

- Test a choice against the facts of the question
- If every fact holds, it is the answer
- If a fact breaks, see which way, and move
- Only arithmetic: easy to see, hard to get wrong

IN THE LESSON

- Here is the idea.
- A question gives you facts and asks for an unknown number.
- One way is to build equations and solve them.
- The other way is to notice that the answer must be one of five numbers, and that checking a number against the facts is usually easy, even when solving for it is hard.
- If a choice satisfies every fact, it is the answer.
- If it breaks a fact, it is wrong, and often the way it breaks tells you which direction to go.
- This is why backsolving is a respectable method and not a trick.
- It uses the information that the question gives you, including the choices.
- It is also hard to get wrong, because you are only doing arithmetic that you can see.

2 Backsolving: the steps

Order the choices, start in the middle

1. Are the choices testable?	numbers, points, intervals	**check**
2. Put them in order	smallest to largest	**sort**
3. Start with the middle	it rules out half the list	**middle**
4. Test against every fact	write what happens	**test**
5. Too big or too small?	move that way, and repeat	**move**

IN THE LESSON

- Here are the steps.
- First, check that the choices are values you can test: numbers, points, or intervals, and not long expressions in letters.
- Second, put the choices in order, from smallest to largest, on your scratch board.
- Third, start with the middle one, not the first.
- The middle choice lets you rule out half of the list at once.
- Fourth, test that choice against every fact in the question, and write down what happens.
- Fifth, if it fails, decide whether the unknown needs to be bigger or smaller, and move to the choice in that direction.
- Sixth, when a choice satisfies every fact, check once more, and stop.
- Start in the middle, and you need at most three tests.

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3 When backsolving shines

- Stories that are awkward to set up but easy to check
- Equations with a parameter, such as which k
- Points on a line; intervals for an inequality
- Plugging in takes seconds; setting up takes thought

IN THE LESSON

- Backsolving is at its best when the setting is a story or an equation that is awkward to set up, but easy to check.
- Rates and times, where one fact is a total over two legs of a journey.
- Ages, and quantities that satisfy two conditions.
- Equations with a parameter, such as finding which value of k gives roots with a special property.
- Questions that ask which point lies on a line, or which interval describes the solutions of an inequality, because you can test a point at a time.
- In each of these, setting up the algebra takes thought, while plugging in a number takes seconds.
- The choices are a gift from the question, and backsolving is how you use them.

4 When not to backsolve

- Choices are expressions: nothing to plug in
- Messy numbers: each test is slow
- Easy algebra: solve it directly
- Check what is asked, and test every fact

IN THE LESSON

- Backsolving has limits.
- If the choices are expressions with letters, there is nothing to plug in, so pick numbers or use algebra instead.
- If the choices are messy numbers, such as fractions with awkward denominators, each test costs time.
- If the question is really quick by algebra, for instance a simple linear equation, solving directly is faster, and it avoids five tests.
- And be careful with questions that ask for something other than the unknown you test.
- If you test a speed but the question asks for minutes, you must convert at the end.
- Always re-read what the question asks for, and finish with a check of the final choice against every fact, not just the last one you tested.

5 Backsolve or set up?

Which is quicker: building it, or testing one choice?

The question	Best method
Neat numbers, easy to check	Backsolve
Which point lies on a line	Backsolve: test each
Which interval is the solution	Backsolve: test points
A short linear equation	Solve directly

IN THE LESSON

- A quick table to decide.
- If the choices are neat numbers, and the facts are easy to check, backsolve.
- If the question says which of the following points lies on a line, or which interval describes the solutions, backsolve by testing.
- If the question asks for a parameter such as k, that makes a special property hold, and you can check that property for a given k, backsolve.
- If the algebra is a single short step, solve it directly.
- If the choices are expressions or algebraic fractions, set up and solve, or pick numbers.
- And when you are not sure, spend ten seconds asking, which is quicker: building the equation, or testing one choice?

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6 Question 1: a point on a line

- Which of the following points lies on the line y = 3x - 5?
- A. (0, 5)
- B. (1, 2)
- C. (1, -12)
- D. (5, 0)
- E. (2, 1)

IN THE LESSON

- Question one is an easy one.
- Read it with me, and notice that every choice is a point you can test.
- Which of the following points lies on the line y equals three x minus five?
- Choice A.
- the point zero, five.
- Choice B.
- the point one, two.
- Choice C.
- the point one, minus twelve.
- Choice D.
- the point five, zero.
- Choice E.
- the point two, one.
- Five points, one rule.
- Test them against the rule.

7 Question 1: test each point

y = 3x - 5: does the y value match?

A: (0, 5)	3 x 0 - 5 = -5, not 5	**no**
B: (1, 2)	3 x 1 - 5 = -2, not 2	**no**
C: (1, -12)	3 x 1 - 5 = -2, not -12	**no**
D: (5, 0)	3 x 5 - 5 = 10, not 0	**no**
E: (2, 1)	3 x 2 - 5 = 1	**yes**

IN THE LESSON

- A point lies on the line when its coordinates make the equation true.
- So test each choice.
- Put the x coordinate into three x minus five, and see whether you get the y coordinate.
- Choice A: x is zero, so three times zero minus five is minus five.
- But the point says y is five.
- No.
- Choice B: x is one, three minus five is minus two, and the point says two.
- No, and notice that this is the right number with the wrong sign.
- Choice C: x is one gives minus two again, and the point says minus twelve.
- No.
- Choice D: x is five, fifteen minus five is ten, and the point says zero.
- No.
- Choice E: x is two, six minus five is one, and the point says one.
- Yes.
- The point two, one lies on the line.

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8 Question 1: the traps in the choices

Substitute, never rearrange

A: (0, 5)	reads the intercept as +5	**lost sign**
B: (1, 2)	right size, wrong sign	**sign slip**
C: (1, -12)	reads 3(x - 5)	**bracket slip**
D: (5, 0)	5 is not the x-intercept	**intercept slip**
E: (2, 1)	3 x 2 - 5 = 1	**the answer**

IN THE LESSON

- Look at what the wrong choices tell us.
- Choice A reads the y intercept as plus five and loses the minus sign.
- Choice B is a point with the right size, minus two swapped for plus two.
- Choice C reads the equation as three times, x minus five, which multiplies the constant as well.
- Choice D takes five as the x intercept, but setting y to zero gives x equal to five thirds.
- Each of these is a slip in the algebra of a line.
- Backsolving avoids them all, because we never rearranged the equation.
- We only substituted.
- The answer is E, the point two, one.
- For a point on a line, testing is nearly always faster than anything else.

9 Question 2: a quadratic with k

- In the equation $x^2 - 12x + k = 0$, one solution is twice the other. What is the value of k?
- A. 16
- B. 32
- C. 36
- D. 48
- E. 72

IN THE LESSON

- Question two is a medium one.
- Read it with me, and think about which choice you would test first.
- In the equation x squared minus twelve x plus k equals zero, one solution is twice the other.
- What is the value of k?
- Choice A.
- sixteen.
- Choice B.
- thirty two.
- Choice C.
- thirty six.
- Choice D.
- forty eight.
- Choice E.
- seventy two.
- The choices are numbers for k.
- Put them in order.

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10 Question 2: start in the middle

$x^2 - 12x + k = 0$: roots add to 12, multiply to k

Order the choices	16, 32, 36, 48, 72	**middle: 36**
k = 36	$(x - 6)^2 = 0$, roots 6 and 6	**not 1 : 2**
Direction	equal roots give the largest k	**go lower**
Next choice down	k = 32	**test it**

IN THE LESSON

- First, order the choices: sixteen, thirty two, thirty six, forty eight, seventy two.
- The middle one is thirty six, so test that first.
- With k equal to thirty six, the equation is x squared minus twelve x plus thirty six, which is x minus six, all squared.
- The only solution is six, repeated.
- Is one solution twice the other?
- Six is not twice six.
- So thirty six fails.
- Which way now?
- The roots always add to twelve.
- When they are equal they are six and six, and their product, which is k, is as big as it can be.
- Spreading the roots apart makes the product smaller.
- So k must be below thirty six, and the next choice down is thirty two.
- Test thirty two next.

11 Question 2: test thirty two

k = 32: two numbers that multiply to 32 and add to 12

k = 32	$x^2 - 12x + 32 = (x - 4)(x - 8)$	**roots 4, 8**
Twice the other?	8 = 2 x 4	**yes**
k = 48 or 72	no real solutions at all	**out**
k = 16	roots 6 plus or minus root 20	**not 1 : 2**

IN THE LESSON

- Test k equal to thirty two.
- The equation is x squared minus twelve x plus thirty two.
- Look for two numbers that multiply to thirty two and add to twelve: four and eight.
- So it factors as x minus four, times x minus eight.
- The solutions are four and eight.
- Is one twice the other?
- Eight is twice four.
- Yes.
- Every fact is satisfied, so k is thirty two.
- Notice we found the roots with no formulas.
- Now glance at the others to be sure.
- For forty eight and seventy two, the numbers under the square root would be negative, so the equation has no real solutions at all.
- For sixteen, the roots are six plus or minus the square root of twenty, which are not in a one to two ratio.
- The answer is B.

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12 Question 2: the traps in the choices

Roots 4 and 8: k is their product

A: 16	squares the smaller root 4	**slip**
B: 32	4 x 8, roots in ratio 1:2	**the answer**
C: 36	6 and 6: ignores 1 : 2	**equal roots**
D: 48	4 x 12: root times the sum	**slip**
E: 72	6 x 12: wrong sum of roots	**slip**

IN THE LESSON

- Now see how the wrong choices were made.
- Choice A, sixteen, squares the smaller root, four, instead of multiplying the two roots.
- Choice C, thirty six, splits the sum into two equal halves of six and ignores the one to two relationship.
- That was our first test, and it was a useful failure, because it told us the direction.
- Choice D multiplies the smaller root by the sum of the roots.
- Choice E writes the sum of the roots as two r, instead of r plus two r, which gives six for r, and multiplies six by twelve.
- The right answer, thirty two, is the product of four and eight.
- The answer is B.
- And if you had set up the algebra directly, calling the roots r and two r, you would have got four and eight too.

13 Question 3: an inequality

- Which of the following describes all real values of x for which |x - 3| < 2x?
- A. x > -3
- B. x > 0
- C. 1 < x < 3
- D. x > 1
- E. x < -3 or x > 1

IN THE LESSON

- Question three is a medium one, and the choices are intervals, not single numbers.
- Read it with me.
- Which of the following describes all real values of x for which the absolute value of x minus three is less than two x?
- Choice A.
- x is greater than minus three.
- Choice B.
- x is greater than zero.
- Choice C.
- x is between one and three.
- Choice D.
- x is greater than one.
- Choice E.
- x is less than minus three, or x is greater than one.
- Intervals can be backsolved too: test one point from each region.

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14 Question 3: test points

|x - 3| < 2x: does the point work?

x = 1/2	|1/2 - 3| = 2.5 against 1	**fails**
Choices with 1/2	A: x > -3, B: x > 0	**out**
x = 4	|4 - 3| = 1 against 8	**works**
Choice C: 1 < x < 3	leaves out 4	**out**

IN THE LESSON

- Here the choices are intervals, so test points, and choose the points that tell the choices apart.
- Choices A and B both include small positive numbers, like one half, and the others do not.
- Test x equal to one half: the left side is the absolute value of one half minus three, which is two and a half.
- The right side is two times a half, which is one.
- Is two and a half less than one?
- No.
- So one half is not a solution, and any choice that includes it is out: A and B.
- Next, x equal to four, which choice C excludes.
- The left side is the absolute value of one, which is one.
- The right side is eight.
- One is less than eight, yes.
- So four is a solution, and choice C, which leaves it out, is wrong.

15 Question 3: the last two points

D against E: test below -3, then the edge

x = -4	|-7| = 7 against -8	**fails**
Choice E includes -4	E: x < -3 or x > 1	**out**
Edge x = 1	2 against 2: not less	**excluded**
x = 2	|-1| = 1 against 4	**works**

IN THE LESSON

- Two choices remain: D, x greater than one, and E, x less than minus three or greater than one.
- They differ on the numbers below minus three.
- Test x equal to minus four.
- The left side is the absolute value of minus four minus three, which is seven.
- The right side is two times minus four, which is minus eight.
- Is seven less than minus eight?
- No.
- So minus four is not a solution, and choice E, which includes it, is out.
- That leaves choice D.
- Do one last check at the edge.
- At x equal to one, the left side is two and the right side is two, and two is not less than two.
- So one is not a solution, which agrees with x strictly greater than one.
- A test at x equal to two works.
- The answer is D.

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16 Question 3: the traps in the choices

Each test point removes the choices that disagree

A: x > -3	one branch only	**half the job**
B: x > 0	only 2x positive	**necessary only**
C: 1 < x < 3	wrongly caps x at 3	**fake bound**
D: x > 1	all tests agree	**the answer**
E: x < -3 or x > 1	squares: lets in -4	**squaring slip**

IN THE LESSON

- Now the traps.
- Choice A solves only one branch of the absolute value, ignoring that the other branch must hold as well.
- Choice B uses only that two x must be positive, which is necessary, but not enough.
- Choice C assumes x must be below three because the absolute value is measured from three, but four was a solution.
- Choice E squares both sides without noticing that two x could be negative, which lets in minus four, where the inequality is false.
- Choice D is the only one that survived every test.
- This is the real power of backsolving with intervals: each test point removes the choices that disagree with it, and the wrong choices were built on algebra slips that algebra would not warn you about.
- The answer is D.

17 Question 4: a boat on a river

- A boat travels 24 km upstream and then 24 km back downstream, taking 5 hours in all. The river's current is 2 km/h. At its own speed in still water, how many minutes would the boat take to travel 15 km?
- A. 90
- B. 75
- C. 112.5
- D. 150
- E. 45

IN THE LESSON

- Question four is a hard one.
- Read it with me, and notice that the choices are minutes, but the facts are about speeds.
- A boat travels twenty four kilometres upstream and then twenty four kilometres back downstream, taking five hours in all.
- The river's current is two kilometres an hour.
- At its own speed in still water, how many minutes would the boat take to travel fifteen kilometres?
- Choice A.
- ninety.
- Choice B.
- seventy five.
- Choice C.
- one hundred twelve point five.
- Choice D.
- one hundred fifty.
- Choice E.
- forty five.
- The choices are minutes for fifteen kilometres.
- What does each one say about the boat's speed?

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18 Question 4: turn each choice into a speed

15 km in each time: what is the still-water speed?

45 minutes = 0.75 hours	15 / 0.75	**20 km/h**
75 minutes = 1.25 hours	15 / 1.25	**12 km/h**
90 minutes = 1.5 hours	15 / 1.5	**10 km/h**
112.5 minutes = 1.875 hours	15 / 1.875	**8 km/h**
150 minutes = 2.5 hours	15 / 2.5	**6 km/h**

IN THE LESSON

- The facts are about the speed of the boat, but the choices are minutes for fifteen kilometres.
- So turn each choice into a still water speed, and then test the facts.
- Order the choices by size: forty five, seventy five, ninety, one hundred twelve and a half, and one hundred fifty.
- Fifteen kilometres in ninety minutes is one and a half hours, so the speed is ten kilometres an hour.
- Seventy five minutes is one and a quarter hours, speed twelve.
- One hundred twelve and a half minutes is one and seven eighths hours, speed eight.
- One hundred fifty minutes, two and a half hours, speed six.
- Forty five minutes, three quarters of an hour, speed twenty.
- The middle choice in size is ninety, which is speed ten.
- Test it first.

19 Question 4: test the middle speed

Speed 10: upstream 8, downstream 12

Upstream	24 km at 8 km/h	**3 hours**
Downstream	24 km at 12 km/h	**2 hours**
Total	3 + 2	**5 hours: yes**
Speed 8 (a neighbour)	4 + 2.4	**6.4: too slow**

IN THE LESSON

- Test the boat speed of ten kilometres an hour, the middle choice.
- Upstream, the current slows it by two, to eight, and twenty four kilometres at eight an hour takes three hours.
- Downstream the current adds two, to twelve, and twenty four kilometres at twelve an hour takes two hours.
- The total is five hours, exactly what the question says.
- Every fact holds.
- So the answer is ninety minutes.
- For safety, glance at a neighbour.
- The choice with speed eight, one hundred twelve and a half minutes, gives an upstream speed of six kilometres an hour, so four hours, and a downstream speed of ten, so two point four hours.
- The total is six point four hours, too slow.
- A slower boat gives a longer trip, so the middle choice was exactly right.

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20 Question 4: the traps in the choices

The choices are results of wrong methods

A: 90	speed 10: 3 + 2 = 5 hours	**the answer**
B: 75	15 km at the 12 downstream	**wrong speed**
C: 112.5	15 km at the 8 upstream	**wrong speed**
D: 150	15 x 10 as minutes	**unit slip**
E: 45	15 km at 20, the two added	**wrong speed**

IN THE LESSON

- See how the traps are built from the numbers in the problem.
- Choice B divides fifteen kilometres by the downstream speed of twelve.
- Choice C divides it by the upstream speed of eight.
- Choice D multiplies fifteen by ten and treats the result as minutes.
- Choice E divides fifteen by twenty, the upstream and downstream speeds added together.
- Each is an easy mistake if you set up the algebra under pressure, and each is a plausible number.
- The algebra for this question needs a quadratic equation, with a negative root to throw away.
- The backsolving took two tests and some arithmetic.
- The answer is A, ninety minutes.
- In your own practice, notice how often the choices are the intermediate results that a wrong method would produce.

21 Accurate and quick

- Order the choices; start in the middle
- A failed test tells you which way to go
- Test every fact, not just the first
- Convert units before testing
- Short algebra? Take it. About two minutes

IN THE LESSON

- Now habits that make backsolving fast.
- First, put the choices in order before you test anything, and start in the middle, so that one test can rule out half of the list.
- Second, when a test fails, ask whether the answer needs to be bigger or smaller, and move that way.
- A failed test is information, not a wasted step.
- Third, test against every fact in the question, not just the first one.
- A choice can satisfy one condition and fail another.
- Fourth, if the choices are in different units from the facts, convert each one before you test, as we did with the boat.
- Fifth, if a question has a short algebraic route, take it.
- Backsolving is a tool, not a rule.
- Sixth, spend about two minutes, and keep your tests in a tidy column.

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22 Quick check

1. The middle choice is too big. Where next?
   Answer: To the smaller choices, and test again.
2. The choices are intervals: how to test?
   Answer: A point that tells the choices apart.
3. Choices in minutes, facts about speed?
   Answer: Convert each choice to a speed first.

IN THE LESSON

- Now a quick check.
- Pause after each question and answer it yourself.
- Question one.
- The choices are five numbers in order, and the middle one fails because it is too big.
- Where do you go next?
- The answer: to the choice below it, or lower, since you need a smaller number.
- Question two.
- The choices are intervals for an inequality.
- How do you test them?
- The answer: pick a point that distinguishes the choices, such as a point in one interval but not another, and test it in the original inequality.
- Question three.
- The choices are in minutes, but the facts are about speed.
- What do you do first?
- The answer: convert each choice to a speed, and then test the facts.

23 Recap: backsolving

- Choices are values: numbers, points, intervals
- Order them; test the middle first
- Test every fact; fail means bigger or smaller
- Convert units; re-read what is asked
- Algebra short? Solve directly

IN THE LESSON

- Let us recap the technique.
- Backsolving uses the five choices as candidate answers.
- Check that the choices are values you can test: numbers, points or intervals.
- Put them in order, and start with the middle one, so one test rules out half of the list.
- Test the choice against every fact in the question, and write down what happens.
- If it fails, work out whether the answer should be bigger or smaller, and move that way.
- Stop when a choice satisfies every fact, and make one last check.
- Convert units so the choice and the facts match, and re-read what the question asks for.
- Skip backsolving when the choices are expressions, or when the algebra is short.

Summary notes for the Charta whiteboard lesson. Figures are from the lesson's made-up examples.

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