Charta | LESSON NOTES

Counting and probability: set it up before you calculate

Quantitative Reasoning

1 Five set-ups

Describe the situation first

Overlapping groups	add, subtract the overlap	**grid**
Arrangements	slots; multiply the choices	**slots**
Selections	does order matter?	**divide or not**
At least one, different	count the opposite	**1 minus**
A repeating game	p = win now + restart x p	**equation**

IN THE LESSON

- Here are the five set-ups.
- For overlapping groups, add the two groups and take away the overlap, which was counted twice.
- Then anyone left over belongs to neither group.
- For arrangements, draw slots, one for each position, and multiply the number of choices for each slot.
- For selections, ask whether order matters.
- If it does not, divide the arrangements by the ways to reorder the chosen items.
- For probability, favourable outcomes over total outcomes, and when the question says at least one, or different, count the opposite and subtract from one.
- And for a game that repeats, let p be the answer, describe one round, and write an equation that says p equals the chance of winning now, plus the chance of starting over, times p.

2 Overlapping groups

- Add the groups, take away the overlap
- At least one = A + B - both
- Neither = total - at least one
- Four parts always add up to the total

IN THE LESSON

- When groups overlap, a Venn diagram or a grid keeps you honest.
- Suppose twenty eight students study French and nineteen study Spanish, and eight study both.
- If you add twenty eight and nineteen you count the eight who study both twice, once in each group.
- So the number who study at least one language is twenty eight plus nineteen minus eight.
- The students who study neither are everyone else, the class size minus those who study at least one.
- A good way to keep track is to split the groups into parts that do not overlap: French only, Spanish only, both, and neither.
- These four parts always add up to the whole class.
- Whenever you are unsure, write the four parts and check that they sum to the total.

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3 Does order matter?

Would swapping two of them be a new outcome?

Situation	Method
Prizes: first, second, third	order matters: 7 x 6 x 5 = 210
A committee of three	order does not: 210 / 6 = 35
Digits of a code	slots; repeats allowed or not
Seating in a row	slots; fewer choices each time

IN THE LESSON

- For arrangements and selections, one question decides the method: does the order matter?
- If the items are placed in different roles, such as first, second and third prize, then swapping two people gives a different outcome, so order matters.
- Draw one slot for each role and multiply the choices.
- If the items are simply chosen as a group, such as a committee, then the order does not matter, so count the ordered arrangements and divide by the number of ways to reorder the group.
- For three items, there are three times two times one, which is six orders.
- So choosing three from seven as a group is two hundred ten divided by six, which is thirty five.
- Always ask, would swapping two of them make a new outcome?

4 At least one, and different: use the complement

- P(event) + P(opposite) = 1
- At least one: 1 - P(none)
- Different: 1 - P(the same)
- Use it when the opposite has fewer cases

IN THE LESSON

- Some probability questions are hard to count directly, but easy to count by the opposite.
- If a question asks for the chance of at least one success, count the chance of no successes, and take it from one.
- If it asks for the chance that two draws are different, count the chance that they are the same, and subtract from one.
- The complement works because the chance of an event and the chance of its opposite always add up to one.
- It is quicker whenever the opposite has fewer cases.
- Two marbles of different colours can happen in many ways, but two of the same colour happens in just a few: both red, both blue or both green.
- Count those, and subtract.

5 A game that repeats

- Let p be the chance our player wins
- One round: win now, lose now, or start over
- p = win now + (start over) x p
- Solve for p: two lines of algebra

IN THE LESSON

- Some probability questions describe a game that could go on for many rounds.
- Adding up an endless list of chances looks impossible, but there is a neat way out.
- Let p be the chance that the player we care about wins.
- Look at one round.
- Either the player wins in that round, or the game ends with the other player winning, or nobody wins and the game starts over from the same position.
- If the game starts over, the player's chance is again p.
- So p equals the chance of winning now, plus the chance of nobody winning times p.
- That is an ordinary equation, with p on both sides, which you solve in two lines.
- Always define a round so that, if the game continues, you are back at the start.

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6 Question 1: two languages

- In a class of 50 students, 28 study French, 19 study Spanish and 8 study both languages. How many students study neither language?
- A. 3
- B. 11
- C. 19
- D. 31
- E. 39

IN THE LESSON

- Question one is an easy one.
- Read it with me, and try it yourself first.
- Notice that the three numbers given are for different groups, and that the question asks about the students who are in neither group.
- In a class of fifty students, twenty eight study French, nineteen study Spanish and eight study both languages.
- How many students study neither language?
- Choice A.
- three.
- Choice B.
- eleven.
- Choice C.
- nineteen.
- Choice D.
- thirty one.
- Choice E.
- thirty nine.
- The question asks for neither, which is the outside of the diagram.

7 Question 1: add, subtract the overlap

50 students; French 28, Spanish 19, both 8

At least one language	28 + 19 - 8	**39**
Neither	50 - 39	**11**
French only, Spanish only	28 - 8 and 19 - 8	**20 and 11**
Check: the four parts	20 + 11 + 8 + 11	**50**

IN THE LESSON

- Count the students who study at least one language.
- French has twenty eight and Spanish has nineteen, which adds to forty seven.
- But the eight who study both were counted in both, so subtract them once.
- Forty seven minus eight is thirty nine.
- So thirty nine students study at least one language.
- The class has fifty students, so those who study neither are fifty minus thirty nine, which is eleven.
- Check with the four parts.
- French only is twenty eight minus eight, which is twenty.
- Spanish only is nineteen minus eight, which is eleven.
- Both is eight.
- Neither is eleven.
- Twenty plus eleven plus eight plus eleven is fifty, the whole class.
- The parts add up, so the answer, eleven, is right.

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8 Question 1: the traps in the choices

Say what each number counts

A: 3	adds 28 and 19, no overlap	**double counted**
B: 11	50 - 39	**the answer**
C: 19	subtracts the 8 twice	**overlap twice**
D: 31	subtracts only the Spanish 19	**one group**
E: 39	at least one, not neither	**wrong target**

IN THE LESSON

- Look at the traps.
- Choice A adds twenty eight and nineteen without removing the eight who were counted twice, which gives forty seven, and fifty minus forty seven is three.
- Choice C subtracts the eight twice, giving thirty one who study a language, and nineteen who study neither.
- Choice D subtracts only the nineteen Spanish students from the class.
- Choice E gives the number who study at least one language, thirty nine, and not the number who study neither.
- After you find a number, always say what it is.
- The answer is B, eleven.
- The four part check, French only, Spanish only, both and neither, would have caught every one of these slips.

9 Question 2: prizes for finalists

- First, second and third prizes are to be awarded to 3 of the 7 finalists in a contest, and no finalist can win more than one prize. In how many different ways can the prizes be awarded?
- A. 21
- B. 35
- C. 210
- D. 343
- E. 5,040

IN THE LESSON

- Question two is an easy one.
- Read it with me.
- Ask yourself whether the order matters.
- First, second and third prizes are to be awarded to three of the seven finalists in a contest, and no finalist can win more than one prize.
- In how many different ways can the prizes be awarded?
- Choice A.
- twenty one.
- Choice B.
- thirty five.
- Choice C.
- two hundred ten.
- Choice D.
- three hundred forty three.
- Choice E.
- five thousand forty.
- Three different prizes, so swapping two winners changes the outcome.

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10 Question 2: slots for the prizes

Slots: first prize, second prize, third prize

First prize	any of the 7 finalists	**7**
Second prize	6 are left	**6**
Third prize	5 are left	**5**
Ways to award the prizes	7 x 6 x 5	**210**
For comparison: a group of 3	210 / 6	**35**

IN THE LESSON

- Draw three slots, one for each prize: first, second and third.
- For the first prize, any of the seven finalists could win.
- Once someone has won, only six remain for the second prize, because no finalist can win more than one prize.
- Then five remain for the third prize.
- Multiply the choices: seven times six times five, which is two hundred ten.
- Order matters here, because the prizes are different.
- Giving first prize to Ann and second to Ben is a different outcome from giving first prize to Ben and second to Ann.
- Compare with choosing a group of three, with no prizes: each group of three would be counted six times, once for each order, so there are two hundred ten divided by six, which is thirty five groups.

11 Question 2: the traps in the choices

Different roles: order matters

A: 21	7 x 3: not an arrangement	**wrong rule**
B: 35	groups of 3: order ignored	**order ignored**
C: 210	7 x 6 x 5	**the answer**
D: 343	7 x 7 x 7: repeats allowed	**winner repeats**
E: 5,040	arranges all 7 finalists	**too many slots**

IN THE LESSON

- Look at the traps.
- Choice A multiplies the seven finalists by the three prizes, which is not a count of anything.
- Choice B counts the groups of three winners and ignores which winner gets which prize, so it forgets that the order matters.
- Choice D allows one finalist to win more than one prize, seven times seven times seven, which the question rules out.
- Choice E arranges all seven finalists in order, seven factorial, though only three prizes are awarded.
- The answer is C, two hundred ten.
- Two questions tell you the method: are the items in different roles, and can they repeat?
- Different roles means order matters, and no repeats means the choices shrink each time.

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12 Question 3: marbles of different colours

- A bag holds 3 red, 4 blue and 5 green marbles. Two marbles are drawn at random without replacement. What is the probability that the two marbles are different colours?
- A. 19/66
- B. 7/12
- C. 47/132
- D. 47/72
- E. 47/66

IN THE LESSON

- Question three is a medium one.
- Read it with me.
- Counting different colours directly has many cases, so think about the opposite.
- A bag holds three red, four blue and five green marbles.
- Two marbles are drawn at random without replacement.
- What is the probability that the two marbles are different colours?
- Choice A.
- nineteen sixty sixths.
- Choice B.
- seven twelfths.
- Choice C.
- forty seven, one hundred thirty seconds.
- Choice D.
- forty seven seventy seconds.
- Choice E.
- forty seven sixty sixths.
- Pairs of different colours, or pairs of the same colour: which has fewer cases?

13 Question 3: count the pairs of the same colour

12 marbles: 66 pairs, all equally likely

All pairs	12 x 11 / 2	**66**
Both red	3 choose 2	**3**
Both blue	4 choose 2	**6**
Both green	5 choose 2	**10**
Same colour	3 + 6 + 10	**19 of 66**

IN THE LESSON

- The bag holds twelve marbles.
- The number of different pairs of marbles is twelve choose two, which is twelve times eleven over two, sixty six.
- Each pair is equally likely, because the draw is at random.
- Pairs of different colours come in three types, red and blue, red and green, blue and green, which is a lot of counting.
- Pairs of the same colour are fewer.
- Both red: three marbles, choose two, which is three pairs.
- Both blue: four marbles, choose two, six pairs.
- Both green: five marbles, choose two, ten pairs.
- In all, three plus six plus ten, which is nineteen pairs of the same colour, out of sixty six.
- So the probability of the same colour is nineteen sixty sixths.

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14 Question 3: the complement, and a check

Different = 1 - same

Same colour	19 / 66	**19/66**
Different colour	1 - 19/66	**47/66**
Check, one draw at a time	(6 + 12 + 20) / 132	**19/66**
Trap: one order only	47 / 132	**half**
Trap: with replacement	1 - 50/144	**47/72**

IN THE LESSON

- Different colours is the opposite of the same colour.
- So the probability of different colours is one minus nineteen sixty sixths, which is forty seven sixty sixths.
- That is choice E.
- Let us check with the draws one at a time.
- The chance that the second marble matches the first is, for red, three twelfths times two elevenths; for blue, four twelfths times three elevenths; for green, five twelfths times four elevenths.
- The top numbers are six, twelve and twenty, which add to thirty eight, over one hundred thirty two, which simplifies to nineteen sixty sixths.
- The same.
- Now the traps.
- If you count only one order of each pair of different colours, you get half, forty seven over one hundred thirty two.
- And if you draw with replacement, you get forty seven over seventy two.

15 Question 3: the traps in the choices

Count the opposite, then subtract from one

A: 19/66	the same colour, not different	**wrong event**
B: 7/12	ignores the second draw	**one draw**
C: 47/132	one order of the pairs only	**half**
D: 47/72	draws with replacement	**wrong model**
E: 47/66	1 - 19/66	**the answer**

IN THE LESSON

- Now the traps.
- Choice A finds the probability that the two marbles are the same colour, which is the opposite of what the question asks.
- Choice B takes the share of marbles that are not green, seven twelfths, ignoring the second draw.
- Choice C counts only one of the two orders in which a pair of different colours can be drawn, so it is half the right answer.
- Choice D draws with replacement, so every draw is from all twelve marbles.
- The question says without replacement.
- The answer is E, forty seven sixty sixths.
- The complement saved us from counting three kinds of different pairs, and the one at a time check agreed.

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16 Question 4: a game that repeats

- Ann and Ben take turns rolling a fair six-sided die, Ann first. Ann wins as soon as she rolls an even number, and Ben wins as soon as he rolls a 6. The game continues until someone wins. What is the probability that Ann wins?
- A. 1/7
- B. 7/12
- C. 5/7
- D. 3/4
- E. 6/7

IN THE LESSON

- Question four is the hard one.
- Read it with me.
- The game could last many rounds, so do not try to list them.
- Ann and Ben take turns rolling a fair six-sided die, Ann first.
- Ann wins as soon as she rolls an even number, and Ben wins as soon as he rolls a six.
- The game continues until someone wins.
- What is the probability that Ann wins?
- Choice A.
- one seventh.
- Choice B.
- seven twelfths.
- Choice C.
- five sevenths.
- Choice D.
- three quarters.
- Choice E.
- six sevenths.
- Let p be the chance that Ann wins.
- Describe one round.

17 Question 4: one round

p = Ann's chance to win the whole game

Ann wins her roll	even number	**1/2**
Ann fails	odd number	**1/2**
Ben fails, given Ann failed	not a 6	**5/6**
Both fail: back to the start	1/2 x 5/6	**5/12**

IN THE LESSON

- Let p be the chance that Ann wins the whole game.
- Describe one round, which is one roll by Ann and then one roll by Ben, if the game is still on.
- Ann rolls first.
- She wins at once if she rolls an even number, which has probability one half.
- Otherwise, with probability one half, she fails, and Ben rolls.
- Ben wins if he rolls a six, with probability one sixth.
- Otherwise, with probability five sixths, he fails too.
- If both fail, the game is back to the very start: Ann to roll, nothing decided.
- The chance that both fail in a round is one half times five sixths, which is five twelfths.
- From that moment, Ann's chance of winning is again p.
- That is the key step.

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18 Question 4: write the equation and solve

p = 1/2 + (5/12) p

The equation	p = 1/2 + (5/12) p	**p on both sides**
Collect the p terms	(7/12) p = 1/2	**simplify**
Solve	p = 1/2 x 12/7	**6/7**
Ben's chance	1 - 6/7	**1/7**

IN THE LESSON

- Now the equation.
- Ann wins the whole game if she wins in this round, with probability one half, or if both fail, with probability five twelfths, and she goes on to win from the start, with probability p.
- So p equals one half plus five twelfths times p.
- Subtract five twelfths of p from both sides: p times one minus five twelfths, which is seven twelfths, equals one half.
- Divide by seven twelfths: p equals one half times twelve sevenths, which is six sevenths.
- Ann wins with probability six sevenths.
- Sense check: Ann rolls first, and her chance of winning a roll is three times Ben's, so she should win most of the time.
- Six sevenths is about eighty six percent, which seems right.
- Ben wins one seventh.

19 Question 4: the traps in the choices

Define the round so that the game restarts

A: 1/7	Ben's chance, not Ann's	**wrong player**
B: 7/12	someone wins in round one	**one round**
C: 5/7	Ben as 1 in 5	**wrong chance**
D: 3/4	odds 1/2 against 1/6	**ignores turns**
E: 6/7	p = 1/2 + 5/12 p	**the answer**

IN THE LESSON

- Now the traps.
- Choice A reports Ben's chance of winning, which is the opposite of Ann's.
- Choice B is the chance that someone wins in the first round, seven twelfths, which is not Ann's chance in the whole game.
- Choice C treats Ben's one in six chance as one in five, and divides one half by one half plus one fifth.
- Choice D compares the two rates as odds, one half against one half plus one sixth, which ignores that Ann rolls first and the turns alternate.
- The answer is E, six sevenths.
- The pattern is useful: whenever a game can repeat, define the position that repeats, and write p in terms of itself.

20 Accurate and quick

- Say what you are counting
- Order question first; draw the slots
- At least one, different: count the opposite
- Repeating game: p = win now + restart x p
- A second route; probability between 0 and 1

IN THE LESSON

- Now habits that keep counting and probability accurate.
- First, say what you are counting before you count.
- Students who study at least one language, or arrangements of three prizes, or pairs of the same colour.
- Second, ask the order question, and write the slots.
- Third, whenever you see at least one or different, count the opposite.
- Fourth, if a game repeats, define p and the restart, and write an equation.
- Fifth, check by a second route if you can, such as the four part check for groups, or the draw by draw check for probability.
- Sixth, simplify fractions at the end, and compare the result with the choices for traps.
- And spend about two minutes a question.
- Seventh, remember that probabilities are between zero and one, which will catch many slips.

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21 Quick check

1. 30 people: 12 tennis, 10 golf, 4 both. Neither?
   Answer: 12: 12 + 10 - 4 = 18; 30 - 18 = 12.
2. A president and a secretary from 5 people?
   Answer: 20: 5 x 4, the roles differ.
3. Two coins: at least one head?
   Answer: 3/4: 1 - 1/4.

IN THE LESSON

- Now a quick check.
- Pause after each question and answer it yourself.
- Question one.
- In a group of thirty, twelve play tennis, ten play golf, and four play both.
- How many play neither?
- The answer: twelve, because twelve plus ten minus four is eighteen play at least one, and thirty minus eighteen is twelve.
- Question two.
- How many ways can a president and a secretary be chosen from five people?
- The answer: twenty, because five times four, the roles differ, so order matters.
- Question three.
- Two fair coins are tossed.
- What is the probability of at least one head?
- The answer: three quarters, because the chance of no heads is one quarter, and one minus a quarter is three quarters.

22 Recap: counting and probability set-ups

- Groups: add, subtract the overlap, check the parts
- Order matters? Slots, or divide by the orders
- At least one, different: the complement
- Repeating game: p on both sides of the equation
- Say what you are counting

IN THE LESSON

- Let us recap the technique.
- For overlapping groups, add the groups, subtract the overlap, and find the outside by subtraction from the total, and check that the four parts add up.
- For arrangements and selections, draw slots, and ask whether order matters: if the roles differ, multiply the choices, and if the items are just a group, divide by the number of orders.
- For probability, count favourable outcomes over total outcomes, and use the complement for at least one, or different.
- Watch out for with and without replacement.
- For a game that repeats, let p be the answer, describe one round, and write an equation in which the restart brings you back to p.
- In every case, say what you are counting.

Summary notes for the Charta whiteboard lesson. Figures are from the lesson's made-up examples.

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