Charta | LESSON NOTES

Estimation: close enough, and quickly

Quantitative Reasoning

1 When estimation is enough

- Look at the choices to see the accuracy needed
- Far apart: a rough estimate settles it
- Close together: estimate, then be more exact
- Closest to, approximately: an estimate is wanted

IN THE LESSON

- Always start by looking at the answer choices, because they tell you how accurate you must be.
- If the choices are one hundred twenty, two hundred forty, four hundred eighty, and so on, each is double the last, so an estimate that is within ten percent settles the question.
- If two choices are close to each other, such as forty seven and forty eight, you need more care, and the estimate only narrows things down.
- The words closest to, approximately and about are the clearest signals that the question wants an estimate.
- But even when the question asks for an exact value, an estimate is a fast way to discard choices that are in the wrong range, before you spend time on the exact work.

2 The steps of a good estimate

Round, compute, compare, and refine only if needed

1. How accurate must I be?	look at the choices	**gap**
2. Friendly numbers	39.8 becomes 40	**round**
3. Do the easy arithmetic	40 x 6 / 2	**120**
4. Compare with the choices	cross out the far ones	**compare**
5. Two left? Refine	improve the key number	**refine**

IN THE LESSON

- Here are the steps.
- First, look at the choices and decide how accurate you need to be.
- Second, replace each awkward number with a friendly one: round thirty nine point eight to forty, two point oh two to two.
- Keep rounding errors small, and try to round some numbers up and some down so the errors cancel.
- Third, do the arithmetic with the friendly numbers, using shortcuts such as multiplying by powers of ten.
- Fourth, compare the result with the choices, and cross out those that are far away.
- Fifth, if two choices remain close, improve the estimate for the numbers that matter most, or compute exactly.
- Sixth, do a sense check: is the answer bigger or smaller than the numbers you started with, and by about the right amount?

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3 Rounding that keeps errors small

Friendly numbers, small errors

Number	Friendly version
39.8 x 6.1 / 2.02	40 x 6 / 2
Up 19.7%, down 20.2%	x 1.2, then x 0.8
3 to the 6th	729, about 730
2^10	1,024, about 1,000

IN THE LESSON

- Rounding well is a skill.
- Round to numbers with one or two significant digits, so the arithmetic is easy.
- For a product or a quotient, the errors from different numbers can cancel: if you round one number up and another down, the result stays close.
- For percent changes, round the percents, so a rise of nineteen point seven percent and a fall of twenty point two percent become twenty and twenty.
- But remember that a rise and a fall of the same percent do not cancel, since the second acts on a different amount.
- For powers, use friendly facts: two to the tenth is about one thousand, and three to the sixth is seven hundred twenty nine.
- And never round so far that you lose the difference between two choices.

4 Spread without a calculation

- Compare each value's distance from the mean
- Values at the mean add nothing to the spread
- Everything far from the mean: the biggest spread
- The range is only the two ends

IN THE LESSON

- Statistics questions on spread rarely need a calculation.
- The standard deviation measures how far the values typically sit from the mean, so you can compare lists by looking at their distances from the mean.
- If every list has the same mean, ask which one has its values furthest from that mean, on average.
- A list with some values exactly at the mean is less spread out.
- A list where the values are all far from the mean, in both directions, is the most spread out.
- Notice that the range, the largest minus the smallest, is not the same thing as the spread.
- A list can have a big range but most of its values close to the middle.
- Compare typical distances, not just the two ends.

5 Large powers

- Simplify with exponent rules first
- Cancel common factors
- Know 2 to the 10th is about 1,000; 3 to the 6th is 729
- Round, then compare with the far-apart choices

IN THE LESSON

- Large powers look frightening, but they are rarely needed exactly.
- First simplify with the rules of exponents.
- A power of six is a power of two times the same power of three, so six to the twelfth is two to the twelfth times three to the twelfth.
- Cancel common factors with the other part of the expression.
- Then estimate what is left, using friendly facts.
- Three to the sixth is seven hundred twenty nine.
- To square that, round to seven hundred thirty, and square the seven, which is forty nine, and put in the zeros: about five hundred thirty thousand.
- Two to the eighth is two hundred fifty six.
- Then divide, rounding to a convenient number, and compare with the choices, which are usually far apart.

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6 Question 1: closest to

- Which of the following is closest to (39.8 × 6.1)/2.02?
- A. 120
- B. 240
- C. 480
- D. 1200
- E. 2400

IN THE LESSON

- Question one is an easy one.
- Read it with me, and notice the two words, closest to.
- The choices are far apart.
- Which of the following is closest to thirty nine point eight times six point one, all divided by two point oh two?
- Choice A.
- one hundred twenty.
- Choice B.
- two hundred forty.
- Choice C.
- four hundred eighty.
- Choice D.
- twelve hundred.
- Choice E.
- twenty four hundred.
- No exact work is needed.
- Round, and compare.

7 Question 1: round and compute

39.8 x 6.1 / 2.02 is about 40 x 6 / 2

Round the numbers	40, 6, 2	**friendly**
Top	40 x 6	**240**
Divide by the bottom	240 / 2	**120**
The exact value, for a check	242.78 / 2.02	**about 120.2**

IN THE LESSON

- Round each number to a friendly one.
- Thirty nine point eight is very close to forty.
- Six point one is close to six.
- Two point oh two is close to two.
- The expression becomes forty times six, divided by two.
- Forty times six is two hundred forty.
- Divided by two is one hundred twenty.
- The nearest choice is one hundred twenty.
- The choices are far apart, doubling each time, so our rough answer easily picks one.
- For a sense check, the exact value is about one hundred twenty point two, since thirty nine point eight times six point one is two hundred forty two point seven eight, and dividing by two point oh two gives one hundred twenty point two.
- The rounding changed the answer by a tiny amount.

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8 Question 1: the traps in the choices

Wrong choices are a factor of 2 or 10 away

A: 120	40 x 6 / 2	**the answer**
B: 240	forgets to divide by 2	**no division**
C: 480	multiplies by 2, not divides	**wrong operation**
D: 1,200	slips a decimal place	**decimal slip**
E: 2,400	decimal slip and no division	**two slips**

IN THE LESSON

- Now the traps.
- Choice B rounds correctly to forty times six, which is two hundred forty, but forgets to divide by two point oh two.
- Choice C multiplies by two instead of dividing, so it doubles two hundred forty, to four hundred eighty.
- Choice D slips a decimal place, treating thirty nine point eight as three hundred ninety eight, before dividing by two.
- Choice E does the same, and also skips the division.
- All four wrong choices are within a factor of ten of the right one, which is the typical range of slips.
- An estimate would have caught them, because each of them is far from what we expect.
- The answer is A, one hundred twenty.
- The estimate was done in a few seconds.

9 Question 2: a price raised, then cut

- A price of $49.80 is raised by 19.7% and the new price is then cut by 20.2%. The final price is closest to which of the following?
- A. $72
- B. $60
- C. $50
- D. $48
- E. $40

IN THE LESSON

- Question two is a medium one.
- Read it with me.
- Two percent changes, each awkward, but each very near a round number.
- A price of forty nine dollars eighty is raised by nineteen point seven percent, and the new price is then cut by twenty point two percent.
- The final price is closest to which of the following?
- Choice A.
- seventy two dollars.
- Choice B.
- sixty dollars.
- Choice C.
- fifty dollars.
- Choice D.
- forty eight dollars.
- Choice E.
- forty dollars.
- Round the percents, and think multipliers.

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10 Question 2: round the multipliers

About $50, up about 20%, then down about 20%

Round the price	49.80	**50**
Round the changes	up 19.7%, down 20.2%	**x 1.2, x 0.8**
Multipliers together	1.2 x 0.8	**0.96**
Final price	50 x 0.96	**48**

IN THE LESSON

- Round everything.
- Forty nine dollars eighty is about fifty dollars.
- A rise of nineteen point seven percent is about twenty percent, which multiplies by one point two.
- A cut of twenty point two percent is about twenty percent, which multiplies by zero point eight.
- So the final price is fifty times one point two times zero point eight.
- Fifty times one point two is sixty.
- Sixty times zero point eight is forty eight.
- Or multiply the multipliers first: one point two times zero point eight is zero point nine six, and fifty times zero point nine six is forty eight.
- The final price is about forty eight dollars.
- Notice the choices: fifty is tempting, because a rise and a fall of twenty percent seem to cancel.
- They do not, since the cut acts on the larger price.

11 Question 2: is the rounding safe?

Closer check: 49.80 x 1.197 x 0.798

After the rise	49.80 x 1.197	**about 59.6**
After the cut	59.6 x 0.798	**about 47.6**
Nearest choice	47.6 is closest to	**48**
Why not 50?	0.96 takes 4% off 50	**2 less**

IN THE LESSON

- Check that the rounding was safe.
- The choices are seventy two, sixty, fifty, forty eight and forty.
- They are spread out, but forty eight and fifty are close, only two dollars apart, so we must be careful.
- The rounding errors went in different directions: nineteen point seven was rounded up, and twenty point two was rounded down, so the errors partly cancel.
- Let us compute more exactly.
- Forty nine dollars eighty times one point one nine seven is about fifty nine dollars sixty.
- Times zero point seven nine eight is about forty seven dollars fifty seven.
- That rounds to forty eight, not fifty.
- The difference between fifty and forty eight comes from the multiplier of zero point nine six, which takes four percent off fifty.
- Four percent of fifty is two.

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12 Question 2: the traps in the choices

A rise and a fall never cancel

A: $72	treats both as rises	**1.2 twice**
B: $60	applies the rise only	**forgot the cut**
C: $50	rise and fall cancel	**no cancel**
D: $48	50 x 1.2 x 0.8	**the answer**
E: $40	applies the cut only	**forgot the rise**

IN THE LESSON

- Now the traps.
- Choice A treats both changes as increases, multiplying by about one point two twice, which is seventy two.
- Choice B applies only the rise, sixty.
- Choice E applies only the cut, forty.
- Both come from forgetting one of the two changes.
- Choice C is the most subtle: it treats a rise of about twenty percent followed by a fall of about twenty percent as cancelling.
- They do not cancel, because the second percentage is taken of a different amount.
- The answer is D, forty eight dollars.
- This is the key fact to remember: a percent rise and an equal percent fall always leave you a little below where you started, by the square of the percent, here four percent.

13 Question 3: which list is most spread out?

- Each of the following lists of six numbers has a mean of 5. Which list has the greatest standard deviation?
- A. 0, 5, 5, 5, 5, 10
- B. 0, 2, 4, 6, 8, 10
- C. 1, 3, 5, 5, 7, 9
- D. 1, 1, 1, 9, 9, 9
- E. 2, 4, 5, 5, 6, 8

IN THE LESSON

- Question three is a medium one on spread.
- Read it with me.
- Do not work out any standard deviations yet.
- Each of the following lists of six numbers has a mean of five.
- Which list has the greatest standard deviation?
- Choice A.
- zero, five, five, five, five, ten.
- Choice B.
- zero, two, four, six, eight, ten.
- Choice C.
- one, three, five, five, seven, nine.
- Choice D.
- one, one, one, nine, nine, nine.
- Choice E.
- two, four, five, five, six, eight.
- All the lists have the same mean, five.
- Compare the distances from five.

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14 Question 3: distances from the mean

Mean 5: how far is each value from 5?

A: 0, 5, 5, 5, 5, 10	5, 0, 0, 0, 0, 5	**mostly at the mean**
B: 0, 2, 4, 6, 8, 10	5, 3, 1, 1, 3, 5	**evenly spread**
C: 1, 3, 5, 5, 7, 9	4, 2, 0, 0, 2, 4	**closer**
D: 1, 1, 1, 9, 9, 9	4, 4, 4, 4, 4, 4	**all far**
E: 2, 4, 5, 5, 6, 8	3, 1, 0, 0, 1, 3	**tightest**

IN THE LESSON

- Every list has a mean of five, so compare how far the numbers sit from five.
- In list A, four of the six values are exactly five, with distance zero.
- Only the zero and the ten are away, by five each.
- In list B the distances are five, three, one, one, three and five: spread out evenly, from near to far.
- In list C they are four, two, zero, zero, two and four.
- In list D every value is one or nine, so every distance is exactly four.
- And in list E the distances are three, one, zero, zero, one and three, all quite small.
- Compare the typical distance.
- List E is clearly the smallest.
- List A has many zeros.
- List D has every value four away, and the values in B and C include distances close to zero.

15 Question 3: confirm with the squared distances

Sum of squared distances from the mean

A	25 + 25	**50**
B	25 + 9 + 1 + 1 + 9 + 25	**70**
C and E	40 for C; 20 for E	**40, 20**
D	6 x 16	**96**
Greatest spread	D has the largest total	**D**

IN THE LESSON

- To be sure, add up the squared distances from the mean, a quick calculation because the distances are small.
- List A: twenty five, plus four zeros, plus twenty five, which is fifty.
- List B: twenty five, nine, one, one, nine and twenty five, which is seventy.
- List C: sixteen, four, zero, zero, four and sixteen, forty.
- List D: six times sixteen, which is ninety six.
- List E: nine, one, zero, zero, one and nine, twenty.
- The lists all have six numbers, so the largest total gives the largest variance, and so the greatest standard deviation.
- That is list D, with ninety six.
- The estimate by distances agreed, and we did not need the square root.
- Notice that list B has the widest range, but not the greatest spread, because its middle values sit close to five.

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16 Question 3: the traps in the choices

Spread is about typical distances, not the range

A: 0, 5, 5, 5, 5, 10	range 10, but four at the mean	**range trap**
B: 0, 2, 4, 6, 8, 10	widest range, evenly spread	**range trap**
C: 1, 3, 5, 5, 7, 9	closer to the mean	**too tight**
D: 1, 1, 1, 9, 9, 9	all values 4 from the mean	**the answer**
E: 2, 4, 5, 5, 6, 8	all close to the mean	**too tight**

IN THE LESSON

- Now the traps.
- Choices A and B are chosen for their range of ten, the largest of the five lists, but the standard deviation is about the typical distance from the mean, and not about the two ends.
- List A has four values sitting exactly at the mean.
- List B has its values spread evenly, so many of them are close to five.
- Choices C and E are picked for their variety of different values, but all their values sit close to the mean.
- List D has no value near the mean at all, which is what makes the spread greatest, even though its range is only eight.
- The answer is D.
- The lesson is that spread is not the same as range, and that a quick look at the distances can settle the question.

17 Question 4: a large power

- Which of the following is closest to $6^{12} / 2^{20}$?
- A. 2,000
- B. 4,000
- C. 8,000
- D. 16,000
- E. 32,000

IN THE LESSON

- Question four is the hard one.
- Read it with me.
- Six to the twelfth is far too big to compute, so simplify first.
- Which of the following is closest to six to the twelfth, divided by two to the twentieth?
- Choice A.
- two thousand.
- Choice B.
- four thousand.
- Choice C.
- eight thousand.
- Choice D.
- sixteen thousand.
- Choice E.
- thirty two thousand.
- Six is two times three.
- What does that do to the powers?

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18 Question 4: simplify the powers first

6^12 = 2^12 x 3^12

Split the base	6^12 = 2^12 x 3^12	**primes**
Cancel the twos	2^12 over 2^20	**1 / 2^8**
What is left	3^12 over 2^8	**simpler**
Know these	2^8 = 256, 3^6 = 729	**facts**

IN THE LESSON

- Do not compute six to the twelfth.
- Use the exponent rules instead.
- Six is two times three, so six to the twelfth is two to the twelfth, times three to the twelfth.
- The expression is two to the twelfth times three to the twelfth, over two to the twentieth.
- Cancel the twos: twenty minus twelve leaves two to the eighth in the denominator.
- So the expression is three to the twelfth over two to the eighth.
- Now the numbers are manageable.
- Two to the eighth is two hundred fifty six.
- Three to the twelfth is three to the sixth, squared.
- Three to the sixth is seven hundred twenty nine.
- We need to square that, and since the choices are far apart, a rough value will do.

19 Question 4: estimate and compare

3^12 is about 530,000; divide by 256

3^6, squared	729 x 729: about 530,000	**530,000**
Divide by 256	530,000 / 256	**about 2,070**
Check	256 x 2,000	**512,000**
Nearest choice	about 2,070	**2,000**

IN THE LESSON

- Estimate three to the twelfth as seven hundred twenty nine squared.
- Round to seven hundred thirty.
- Seven times seven is forty nine, and seven hundred thirty squared is about five hundred thirty thousand.
- The exact value is five hundred thirty one thousand, four hundred forty one.
- Now divide by two hundred fifty six.
- Five hundred thirty thousand over about two hundred fifty six is about two thousand and seventy.
- A quick check: two hundred fifty six times two thousand is five hundred twelve thousand, just under five hundred thirty.
- So the answer is a little over two thousand, and the nearest choice is two thousand.
- The other choices are double, four times, eight times and sixteen times that, so there is no confusion.
- The answer is A.

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20 Question 4: the traps in the choices

Each miscount of the twos doubles the answer

A: 2,000	3^12 over 2^8	**the answer**
B: 4,000	leaves 2^7 below	**cancel slip**
C: 8,000	leaves 2^6 below	**cancel slip**
D: 16,000	leaves 2^5 below	**cancel slip**
E: 32,000	leaves 2^4 below	**cancel slip**

IN THE LESSON

- Look at the structure of the choices.
- They are two thousand, then four, eight, sixteen and thirty two thousand: each is double the last.
- That means the wrong choices come from cancelling twelve factors of two, but leaving the denominator as two to the seventh, two to the sixth, two to the fifth or two to the fourth, instead of two to the eighth.
- Each missing factor of two doubles the answer.
- The key skill is counting the factors of two carefully: twenty minus twelve is eight.
- If your count is off by one, your answer is off by a factor of two, and the choices are built to catch you.
- The answer is A, two thousand.
- Once the exponent is right, the rough estimate is enough.

21 Accurate and quick

- Look at the choices: how accurate must I be?
- Friendly numbers; round up and down
- Keep track of the size: digits, powers of ten
- Estimate to check every exact answer
- Simplify first: exponents, common factors

IN THE LESSON

- Now habits that make estimation reliable.
- First, look at the choices before you start, to see how accurate you must be.
- Second, round to friendly numbers, and round some up and some down.
- Third, keep track of the size of the answer: use powers of ten, and say how many digits you expect.
- Fourth, when the choices are close, improve only the number that matters most, or compute exactly.
- Fifth, use estimation as a check on every exact answer, because it catches slips of a factor of ten, an inverted fraction and a missed step.
- Sixth, simplify first, with exponents and common factors, because that is often an estimate in itself.
- And spend about two minutes a question.
- Estimating well is not guessing.
- It is knowing how wrong you are allowed to be.

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22 Quick check

1. About 81.9 times 9.8?
   Answer: About 800: 80 x 10.
2. Up 30%, then down 30%: above or below the start?
   Answer: Below: 1.3 x 0.7 = 0.91.
3. Same mean: values close, or far from the mean?
   Answer: Far from the mean: the larger standard deviation.

IN THE LESSON

- Now a quick check.
- Pause after each question and answer it yourself.
- Question one.
- What is about eighty one point nine, times nine point eight?
- The answer: about eight hundred, because eighty times ten is eight hundred.
- Question two.
- A price goes up thirty percent and then down thirty percent.
- Is the result above or below the start?
- The answer: below, because one point three times zero point seven is zero point nine one, a nine percent fall.
- Question three.
- Two lists have the same mean.
- List one has all its values close to the mean.
- List two has values far from the mean.
- Which has the larger standard deviation?
- The answer: list two, because its values are far from the mean.

23 Recap: estimation

- Choices set the accuracy needed
- Friendly numbers; errors cancel
- Rise and fall of the same percent: not cancelling
- Spread: distances from the mean, not the range
- Powers: simplify, then use friendly facts

IN THE LESSON

- Let us recap the technique.
- Look at the choices to see how accurate you must be.
- Replace awkward numbers with friendly ones, rounding some up and some down so the errors cancel.
- Do the easy arithmetic, keep track of the size of the answer, and compare with the choices.
- If two choices are close, improve the key number, or compute exactly.
- For percent changes, use multipliers, and remember that a rise and a fall of the same percent do not cancel.
- For spread, compare the distances from the mean, rather than the range.
- For large powers, simplify with the exponent rules first, and then use friendly facts such as three to the sixth equals seven hundred twenty nine.
- And estimate to check every exact answer.

Summary notes for the Charta whiteboard lesson. Figures are from the lesson's made-up examples.

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