Charta | LESSON NOTES

Working without a calculator: let the structure do the work

Quantitative Reasoning

1 Why no calculator is not a problem

- Test numbers are chosen to be tidy
- About to multiply big numbers? You missed a trick
- The skill is spotting structure, not arithmetic
- Ten seconds of looking before any calculating

IN THE LESSON

- A calculator is only useful if the numbers are ugly, and the test writers do not set questions with ugly numbers on purpose.
- Instead they set questions where the numbers are tidy once you see the trick.
- Look at the sizes: a few digits, small powers, fractions with small denominators.
- If you find yourself about to multiply two five digit numbers, stop.
- You have missed a simpler route.
- The real skill being tested is not arithmetic.
- It is spotting structure: a common factor, a cancelling pair, a power of ten, a pattern in a product.
- So the rule for this lesson is simple.
- Before you calculate, spend ten seconds looking at the numbers, and ask what makes them special.

2 Four moves that replace the calculator

Look for structure before you calculate

1. Move the decimal point	shift both numbers equally	**whole numbers**
2. Group coefficients and powers	numbers first, then exponents	**separate**
3. Factor out a common power	take out the smaller power	**small sum**
4. Cancel before multiplying	write the factors out	**cross out**

IN THE LESSON

- Here are the four moves.
- First, move the decimal point.
- A division by a decimal becomes a division by a whole number if you shift both numbers the same number of places.
- Second, group the numbers and the powers of ten separately.
- In scientific notation, deal with the coefficients, then with the exponents.
- Third, factor.
- If an expression has the same base raised to different powers, take out the smaller power, and the rest is small arithmetic.
- Fourth, cancel before you multiply.
- In a product of fractions, write the factors out, and cross out what appears on the top and on the bottom.
- In every case you do far less arithmetic than the direct route, and the arithmetic you do is easy to check.

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3 Decimal and power of ten rules

Rules for decimals and exponents

Situation	Move
Divide by a decimal	shift both points the same
Multiply decimals	multiply, then count places
Multiply powers, same base	add the exponents
Divide powers, same base	subtract the exponents

IN THE LESSON

- Keep a few rules at your fingertips.
- To divide by a decimal, shift the decimal point in both numbers the same number of places until the divisor is whole.
- Four tenths divided by two hundredths is the same as forty divided by two.
- To multiply decimals, multiply the numbers as if they were whole, then put the decimal point back by counting the places in both numbers.
- For powers of ten, when you multiply, add the exponents, and when you divide, subtract them, and remember that dividing by ten to the minus four is the same as multiplying by ten to the four.
- And for any base, the same rules hold: a power of five times a power of five adds the exponents, and a power divided by a power subtracts them.

4 Factoring instead of calculating

- Same base, different powers: take out the smaller
- 5 to the x+2 is 5 to the x times 25
- Leaves small numbers: 25 - 1 = 24
- Works for sums of powers too

IN THE LESSON

- The third move deserves its own slide.
- Suppose you meet five to the power x plus two, minus five to the power x.
- You could try to work out x, but notice that both terms contain five to the power x.
- Five to the x plus two is five to the x, times five squared.
- So the expression is five to the x, times twenty five, minus five to the x, which is five to the x times twenty four.
- That is factoring: taking out the common power, and being left with small numbers.
- The same move works for sums such as three to the ten, plus three to the eleven, plus three to the twelve.
- Take out three to the ten, and what is left is one plus three plus nine.
- Look for the smallest power, and take it out.

5 Cancelling in a long product

- Rewrite each factor to expose a pattern
- Write the first few and the last few terms
- Top cancels with a neighbour's bottom
- Only the first and last pieces survive

IN THE LESSON

- The fourth move is for long products of fractions.
- A product with many factors looks horrible, but the factors are usually built from a pattern, and the pattern cancels.
- The trick is to rewrite each factor, and to write the first few and the last few, so you can see which terms meet.
- For instance, a factor of the form one minus one over k squared can be written as k minus one over k, times k plus one over k.
- Multiply a chain of such factors, and the top of each fraction cancels with the bottom of a neighbour, so almost everything vanishes, leaving just the first and last pieces.
- This is called telescoping, like a collapsing telescope.
- Always write enough terms to see the pattern, and then trust it.

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6 Question 1: dividing decimals

- What is the value of 0.48 ÷ 0.006?
- A. 0.08
- B. 0.8
- C. 8
- D. 80
- E. 800

IN THE LESSON

- Question one is an easy one.
- Read it with me, and try it yourself first.
- Notice that both numbers are decimals, and the one you divide by is the smaller of the two.
- What is the value of zero point four eight, divided by zero point zero zero six?
- Choice A.
- zero point zero eight.
- Choice B.
- zero point eight.
- Choice C.
- eight.
- Choice D.
- eighty.
- Choice E.
- eight hundred.
- Dividing by a decimal: let us make the divisor a whole number.

7 Question 1: shift both decimal points

Multiply both numbers by 1,000

Divisor 0.006	shift three places	**6**
Dividend 0.48	shift three places	**480**
New division	480 / 6	**80**
Sense check	dividing by less than 1 grows	**80 is big**

IN THE LESSON

- The divisor is zero point zero zero six.
- To make it a whole number, shift its decimal point three places to the right, which gives six.
- We must treat the other number in the same way, or we would change the answer.
- Shift the decimal point in zero point four eight three places to the right too: it becomes four hundred eighty.
- The value of the division has not changed, because multiplying both numbers by one thousand cancels out.
- So the question is four hundred eighty divided by six.
- Six goes into forty eight eight times, so six goes into four hundred eighty eighty times.
- The answer is eighty.
- A quick sense check: dividing by a number less than one makes the answer bigger, so the answer must be larger than zero point four eight.
- Eighty is.

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8 Question 1: the traps in the choices

Both points must shift by the same amount

A: 0.08	treats 0.006 as 6	**one shift only**
B: 0.8	treats 0.006 as 0.6	**two places**
C: 8	shifts 0.48 only two places	**uneven shift**
D: 80	480 divided by 6	**the answer**
E: 800	shifts 0.48 four places	**uneven shift**

IN THE LESSON

- Look at the traps, because every one is a shifting error.
- Choice A treats zero point zero zero six as six, ignoring its decimal places, and so computes zero point four eight divided by six.
- Choice B treats it as zero point six, dropping two of its places.
- Choice C shifts the divisor three places but the dividend only two.
- Choice E shifts the dividend four places, one more than the divisor.
- The rule behind all of them is the same: the two decimal points must move by the same number of places.
- The answer is D, eighty.
- When your answer is far from a trap such as eight or eight hundred, check that you counted the places correctly.
- The sense check, that dividing by a small number makes a big number, would have ruled out A and B immediately.

9 Question 2: scientific notation

- What is the value of $(1.2 \times 10^{5} \times 3 \times 10^{-3}) \div (9 \times 10^{-4})$?
- A. 4,000,000
- B. 400,000
- C. 40,000
- D. 360
- E. 0.324

IN THE LESSON

- Question two is a medium one.
- Read it with me.
- Three numbers and three powers of ten.
- Do not multiply them in one go.
- What is the value of, one point two times ten to the five, times three times ten to the minus three, all divided by nine times ten to the minus four?
- Choice A.
- four million.
- Choice B.
- four hundred thousand.
- Choice C.
- forty thousand.
- Choice D.
- three hundred sixty.
- Choice E.
- zero point three two four.
- Deal with the coefficients first, then with the powers of ten.

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10 Question 2: coefficients, then powers

Coefficients: 1.2, 3, 9. Powers: 10 to 5, -3, -4

Coefficients	1.2 x 3 = 3.6; 3.6 / 9 = 0.4	**0.4**
Powers of ten	5 + (-3) - (-4)	**6**
Powers combine	10 to the 6	**1,000,000**
Value	0.4 x 1,000,000	**400,000**

IN THE LESSON

- Split each number into a coefficient and a power of ten, and handle them separately.
- The coefficients are one point two, three, and nine.
- One point two times three is three point six.
- Three point six divided by nine is zero point four.
- Now the powers of ten.
- Ten to the five, times ten to the minus three: add the exponents, five and minus three, which makes ten squared.
- Then divide by ten to the minus four: subtract the exponent, so two minus minus four, which is six.
- So the powers combine to ten to the six, which is one million.
- The whole value is zero point four times one million, which is four hundred thousand.
- The answer is B.
- A sense check: the top is about three hundred sixty, and we divide by a very small number, nine ten thousandths, so the result must be huge.

11 Question 2: a second route to check

Top first, then the bottom

Top	120,000 x 3 / 1,000	**360**
Divide by 9	360 / 9	**40**
Divide by 10 to the -4	multiply by 10,000	**400,000**
Trap: stop at the top	360	**choice D**

IN THE LESSON

- Check by a second route.
- First work out the top.
- One point two times ten to the five is one hundred twenty thousand.
- Times three times ten to the minus three: three thousandths, so one hundred twenty thousand times three is three hundred sixty thousand, divided by one thousand, which is three hundred sixty.
- So the top is three hundred sixty.
- The bottom is nine times ten to the minus four, which is nine ten thousandths.
- Dividing by nine ten thousandths is the same as dividing by nine and multiplying by ten thousand.
- Three hundred sixty divided by nine is forty.
- Forty times ten thousand is four hundred thousand.
- Both routes agree.
- Notice that the second route showed us the trap in choice D: three hundred sixty is the top on its own, which is what you get if you forget to divide.

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12 Question 2: the traps in the choices

Most slips are one power of ten out

A: 4,000,000	3.6 / 9 as 4, not 0.4	**ten times too big**
B: 400,000	0.4 x 10 to the 6	**the answer**
C: 40,000	loses a power of ten	**10 times too small**
D: 360	the top only	**forgot to divide**
E: 0.324	multiplies by the bottom	**wrong operation**

IN THE LESSON

- Now the traps.
- Choice A treats three point six divided by nine as four instead of zero point four, so it is ten times too big.
- Choice C takes ten to the five, times ten to the minus three, divided by ten to the minus four as ten to the five instead of ten to the six, so it is ten times too small.
- Both are errors of one power of ten, which is the most common kind in scientific notation.
- Choice D is the top on its own, three hundred sixty.
- Choice E multiplies by the denominator, nine times ten to the minus four, instead of dividing by it.
- The answer is B, four hundred thousand.
- When your answer differs from a choice by a factor of ten, count your exponents again.

13 Question 3: factoring powers

- If 5^(x + 2) − 5^x = 600, what is the value of 5^(x + 1)?
- A. 625
- B. 125
- C. 25
- D. 24
- E. 5

IN THE LESSON

- Question three is a medium one.
- Read it with me.
- Resist the urge to solve for x.
- If five to the power x plus two, minus five to the power x, equals six hundred, what is the value of five to the power x plus one?
- Choice A.
- six hundred twenty five.
- Choice B.
- one hundred twenty five.
- Choice C.
- twenty five.
- Choice D.
- twenty four.
- Choice E.
- five.
- What do the two terms have in common?

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14 Question 3: take out the common power

5 to the x+2 minus 5 to the x = 600

Rewrite	5 to the x+2 = 25 x 5 to the x	**25 times**
Factor	(25 - 1) x 5 to the x	**24 x 5 to the x**
Solve	24 x 5 to the x = 600	**5 to the x = 25**
The question asks	5 to the x+1 = 5 x 25	**125**

IN THE LESSON

- Both terms on the left contain five to the x.
- Five to the power x plus two equals five to the x, times five squared, which is twenty five times five to the x.
- So the left side is twenty five times five to the x, minus five to the x.
- That is twenty five minus one, times five to the x, which is twenty four times five to the x.
- The equation says this equals six hundred.
- So five to the x equals six hundred divided by twenty four, which is twenty five.
- We have found five to the x.
- But the question asks for five to the power x plus one, which is five to the x, times five.
- That is twenty five times five, which is one hundred twenty five.
- We never had to find x, though we could see it is two, since five squared is twenty five.

15 Question 3: the traps in the choices

Re-read which power is asked for

A: 625	5 to the x+2, the first term	**wrong power**
B: 125	5 x 25	**the answer**
C: 25	5 to the x, not x+1	**wrong power**
D: 24	the factor 25 - 1	**stopped early**
E: 5	5 to the x-1 when x is 2	**wrong power**

IN THE LESSON

- Look at the traps.
- They are almost all about which power the question wants.
- Choice A is five to the x plus two, which is six hundred twenty five.
- Choice C is five to the x, which is twenty five, the value we found on the way.
- Choice E is five to the x minus one, which is five.
- Choice D is the factor twenty four, which comes from stopping after the factoring and before the solving.
- Each is a number that appears naturally during the work, and the question asks for one of them.
- After you find a quantity, always re-read what the question asked for.
- The answer is B, one hundred twenty five.
- And the whole problem used only the numbers five, twenty five and twenty four.

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16 Question 4: a long product

- What is the value of the product (1 - 1/4)(1 - 1/9)(1 - 1/16) ... (1 - 1/100), in which the factors are 1 - 1/k² for each integer k from 2 to 10?
- A. 9/10
- B. 11/20
- C. 1/2
- D. 10/21
- E. 9/20

IN THE LESSON

- Question four is the hard one.
- Read it with me.
- Nine factors.
- Do not multiply them one at a time.
- What is the value of the product, one minus one quarter, times one minus one ninth, times one minus one sixteenth, and so on, up to one minus one over one hundred, in which the factors are one minus one over k squared for each integer k from two to ten?
- Choice A.
- nine tenths.
- Choice B.
- eleven twentieths.
- Choice C.
- one half.
- Choice D.
- ten twenty firsts.
- Choice E.
- nine twentieths.
- Look at one factor: how could you rewrite one minus one over k squared?

17 Question 4: rewrite each factor

1 - 1/k² = (k - 1)/k x (k + 1)/k

k = 2	1/2 x 3/2	**3/4**
k = 3	2/3 x 4/3	**8/9**
k = 4	3/4 x 5/4	**15/16**
Pattern	(k - 1)/k x (k + 1)/k	**two chains**

IN THE LESSON

- Start with one factor.
- One minus one over k squared is k squared minus one, over k squared.
- That factors as k minus one, times k plus one, over k times k.
- So each factor is two fractions multiplied: k minus one over k, and k plus one over k.
- Write the first few factors this way.
- For k equal to two: one half, times three halves.
- For k equal to three: two thirds, times four thirds.
- For k equal to four: three quarters, times five quarters.
- Now see the pattern.
- The first fractions are one half, two thirds, three quarters, and so on.
- The second fractions are three halves, four thirds, five quarters, and so on.
- Each group is a chain, and each chain cancels.

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18 Question 4: cancel the chains

Product of the chains: 1/10 times 11/2

Chain 1: 1/2 x 2/3 x ... x 9/10	first top over last bottom	**1/10**
Chain 2: 3/2 x 4/3 x ... x 11/10	last top over first bottom	**11/2**
Multiply the chains	1/10 x 11/2	**11/20**
Quick check	11/20 is a little over a half	**plausible**

IN THE LESSON

- Take the first chain, the k minus one over k fractions, from k equal to two up to ten: one half, times two thirds, times three quarters, and so on, up to nine tenths.
- The numerator of each cancels the denominator of the one before it, so only the first numerator, one, and the last denominator, ten, survive.
- That chain equals one tenth.
- The second chain, the k plus one over k fractions: three halves, times four thirds, times five quarters, up to eleven tenths.
- Here the denominator of each cancels the numerator of the one before.
- Only the first denominator, two, and the last numerator, eleven, survive.
- That chain is eleven halves.
- Multiply the two results: one tenth times eleven halves is eleven twentieths.
- That is the answer, B.

19 Question 4: the traps in the choices

The chain leaves (n + 1) / 2n

A: 9/10	only the last factor of chain 1	**no cancelling**
B: 11/20	(n + 1) / 2n with n = 10	**the answer**
C: 1/2	the limit, not the product	**too early**
D: 10/21	n / (2n + 1)	**wrong formula**
E: 9/20	(n - 1) / 2n	**off by one**

IN THE LESSON

- Now the traps.
- Choice A takes only the last factor of the first chain, nine tenths, and ignores the other chain and the cancelling.
- Choice C is the value the infinite product approaches, one half, but the finite product stops short of it, so it is a little larger.
- Choices D and E telescope correctly but end with the wrong formula, one with n over two n plus one, and the other with n minus one over two n.
- The right formula is n plus one over two n, which for n equal to ten is eleven twentieths.
- The check is easy: the first three factors multiply to five eighths, which is already close to one half, and each further factor is close to one, so a result a little over one half is expected.
- The answer is B.

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20 Accurate and quick

- Look before you calculate
- Sense check: bigger or smaller? how many digits?
- Group coefficients and powers in two columns
- One power of ten out? Recount the exponents
- Long arithmetic means a missed trick

IN THE LESSON

- Now habits that keep you fast without a calculator.
- First, look before you calculate.
- Ten seconds of looking at the shape of the numbers often removes a minute of arithmetic.
- Second, use a sense check on every answer.
- Is the result bigger or smaller than the numbers you started with?
- Does it have about the right number of digits?
- Third, keep the work tidy: group coefficients and powers in two columns, and write out the first few factors of a product.
- Fourth, check any answer that is a factor of ten away from a choice, because a slip of one power of ten is the commonest error.
- Fifth, re-read what is asked for after you find a value.
- Sixth, spend about two minutes, and if arithmetic starts to look long, you are missing a trick.

21 Quick check

1. 0.9 divided by 0.03?
   Answer: Shift both two places: 90 / 3 = 30.
2. 1/2 x 2/3 x 3/4 x ... x 9/10?
   Answer: 1/10: only the first top and last bottom survive.
3. 3 to the 10 plus 3 to the 11?
   Answer: 3 to the 10 times (1 + 3), which is 4 times 3 to the 10.

IN THE LESSON

- Now a quick check.
- Pause after each question and answer it yourself.
- Question one.
- How do you work out zero point nine divided by zero point zero three?
- The answer: shift both decimal points two places, giving ninety divided by three, which is thirty.
- Question two.
- A product of fractions has the form one half, times two thirds, times three quarters, up to nine tenths.
- What is it?
- The answer: one tenth, because everything cancels except the first top and the last bottom.
- Question three.
- Three to the ten, plus three to the eleven, has a common factor.
- What is the sum in terms of three to the ten?
- The answer: three to the ten times four, because one plus three is four.

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22 Recap: working without a calculator

- Divide by a decimal: shift both points the same
- Scientific notation: coefficients, then exponents
- Same base: factor out the smaller power
- Long products: rewrite, then cancel the chains
- Sense check, then re-read the question

IN THE LESSON

- Let us recap the four moves.
- Move the decimal point: when you divide by a decimal, shift both numbers by the same number of places until the divisor is whole.
- Group coefficients and powers of ten separately in scientific notation, add exponents when multiplying, and subtract when dividing.
- Factor out the smaller power when the same base appears with different exponents, so that only small numbers remain.
- And in a long product of fractions, rewrite each factor, write the first and last few, and cancel the chains, so that only the ends survive.
- Always do a sense check on the answer, and re-read the question to make sure you give the quantity it asks for.
- If the arithmetic looks long, you have missed a trick.

Summary notes for the Charta whiteboard lesson. Figures are from the lesson's made-up examples.

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