Charta | LESSON NOTES

Number properties: check, do not guess

Quantitative Reasoning

1 Four checks that solve most questions

- Small numbers: test one or two that obey the facts
- Rules: odd and even, remainders
- Prime factors: write each number as a product of primes
- Systematic counts: no gaps, no doubles

IN THE LESSON

- Here is the kit.
- The first check is the small number test.
- Pick one or two small numbers that obey the facts, and work out the answer.
- It will not prove the general case, but it will tell you which choices are wrong.
- The second check is the rule check.
- A handful of rules, such as odd plus even is odd, settle every odd and even question without any numbers.
- The third check is the prime factor check.
- Write each number as a product of primes, and questions on factors, multiples, common divisors and the count of divisors become reading exercises.
- And the fourth check is the systematic count, such as listing in order, or inclusion and exclusion, so nothing is missed or counted twice.

2 The odd and even rules

The rules settle odd and even questions

Operation	Result
even + even, odd + odd	even
odd + even	odd
any product with an even factor	even
odd x odd, odd squared	odd
2 x anything	even

IN THE LESSON

- Learn these rules cold, and a large family of questions becomes mechanical.
- Even plus even is even.
- Odd plus odd is even.
- Odd plus even is odd.
- For products: any product with an even factor is even, and only odd times odd is odd.
- For powers, a square of an odd number is odd, and a square of an even number is even.
- And any number times two is even, whatever the number is.
- This last one is the most useful, because it hides inside expressions like x plus two y, where two y is always even, so the whole expression is odd when x is odd.
- When a question says which of the following must be odd, check each choice with these rules, one at a time.

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3 Remainders: write the number out

Divided by 7 with remainder 4: n = 7k + 4

Possible n	4, 11, 18, 25, ...	**always 4 more**
Write it out	n = 7k + 4	**algebra**
A remainder must be small	0 up to one less than 7	**reduce it**
Work on remainders	add or multiply, then reduce	**shortcut**

IN THE LESSON

- Remainder questions have a simple secret.
- If a number n leaves a remainder of four when divided by seven, then n is seven times some whole number, plus four.
- So n could be four, or eleven, or eighteen, or twenty five: always four more than a multiple of seven.
- To answer a question about n, pick one such number and do the arithmetic, or write n as seven k plus four and work with that.
- A remainder always sits between zero and one less than the divisor, so a remainder of twelve when dividing by seven is not finished: take out one seven, and the remainder is five.
- Remainders behave like the numbers themselves under adding and multiplying, so you can work on the remainders alone, and reduce at the end.

4 Factors, multiples and divisors

- Break every number into primes
- Common factors: the smaller power of each prime
- Divisor count: add one to each exponent, multiply
- Common divisors are the divisors of the GCF

IN THE LESSON

- For factors and multiples, break every number into primes.
- Then the common divisors of two numbers are the numbers built from their shared primes, using the smaller power of each.
- That product is the greatest common factor.
- The least common multiple takes the larger power of each prime.
- To count the divisors of a number, write it as primes with exponents, add one to each exponent, and multiply.
- For example, a number that is two cubed times three squared has four times three, which is twelve, divisors.
- The extra one is for the choice of using none of that prime.
- And a number divides two numbers exactly when it divides their greatest common factor, so counting common divisors is just counting the divisors of the greatest common factor.

5 Question 1: odd and even

- If x is an odd integer and y is an even integer, which of the following must be odd?
- A. x + 2y
- B. xy
- C. x + y + 1
- D. 2x + y
- E. x^2 - 1

IN THE LESSON

- Question one is an easy one.
- Read it with me, and try it yourself before I solve it.
- If x is an odd integer and y is an even integer, which of the following must be odd?
- Choice A.
- x plus two y.
- Choice B.
- x times y.
- Choice C.
- x plus y plus one.
- Choice D.
- two x plus y.
- Choice E.
- x squared minus one.
- The key word is must: it has to be odd for every odd x and every even y.

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6 Question 1: test small numbers

x odd, y even: pick x = 3, y = 4, then x = 5, y = 2

A: x + 2y	11, then 9	**odd, odd**
B: xy	12, then 10	**even**
C: x + y + 1	8, then 8	**even**
D: 2x + y	10, then 12	**even**
E: x² - 1	8, then 24	**even**

IN THE LESSON

- Start with the small number test.
- Pick x equal to three, which is odd, and y equal to four, which is even.
- Now test each choice.
- Choice A: three plus two times four is eleven.
- Odd.
- Choice B: three times four is twelve.
- Even.
- Choice C: three plus four plus one is eight.
- Even.
- Choice D: two times three plus four is ten.
- Even.
- Choice E: three squared minus one is eight.
- Even.
- Only choice A came out odd.
- That is promising, but must means every case.
- Try a second pair, x equal to five and y equal to two.
- Choice A is five plus four, nine.
- Odd again.
- The others: ten, eight, twelve and twenty four.
- All even.
- So choice A is odd both times, and the other four failed at least once.

7 Question 1: the rules explain why

Rules: 2 x anything is even; odd + even is odd

A: x + 2y	odd + even	**odd**
B: xy	an even factor	**even**
C: x + y + 1	odd + even + 1	**even**
D: 2x + y	even + even	**even**
E: x² - 1	odd - 1	**even**

IN THE LESSON

- Now confirm with the rules, because a check by number is only evidence.
- Choice A: two y is two times anything, so it is even, and odd plus even is odd.
- This will always be odd.
- Choice B: x times y has an even factor, so it is even.
- Choice C: odd plus even is odd, and then plus one makes it even.
- Choice D: two x is even, and y is even, so even plus even is even.
- Choice E: an odd number squared is odd, and odd minus one is even.
- So the rules prove that A is always odd, and the other four are always even.
- The number test and the rule check agree, and we could use either one alone.
- In an exam, the number test is the faster way to find the answer, and the rules are the faster way to be certain.

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8 Question 1: the traps in the choices

Each wrong choice breaks one rule

A: x + 2y	odd + even is odd	**the answer**
B: xy	odd x even is even	**product slip**
C: x + y + 1	adds 1 after odd + even	**off by one**
D: 2x + y	2x is even whatever x is	**missed the 2**
E: $x^2 - 1$	odd squared is odd	**square slip**

IN THE LESSON

- Look at the traps.
- Choice B assumes that odd times even is odd, but any product with an even factor is even.
- Choice C correctly finds that x plus y is odd, and then forgets that adding one changes odd to even.
- Choice D overlooks that two x is even whatever x is, so two x plus y is even plus even.
- Choice E treats x squared as even, but the square of an odd number is odd, so x squared minus one is even.
- Each wrong choice is a slip in one of the rules.
- The answer is A.
- The best habit for odd and even questions is to test each choice with one example, and then confirm the survivor with the rules.

9 Question 2: remainders

- When the positive integer n is divided by 7, the remainder is 4. What is the remainder when 3n is divided by 7?
- A. 0
- B. 1
- C. 4
- D. 5
- E. 12

IN THE LESSON

- Question two is an easy one, and a good one for the small number test.
- Read it with me.
- Notice that the question says nothing about how big n is, only about what is left over when you divide it by seven.
- When the positive integer n is divided by seven, the remainder is four.
- What is the remainder when three n is divided by seven?
- Choice A.
- zero.
- Choice B.
- one.
- Choice C.
- four.
- Choice D.
- five.
- Choice E.
- twelve.
- Write down a few numbers that leave remainder four on division by seven.

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10 Question 2: pick numbers that fit

n leaves remainder 4 on division by 7: remainder of 3n?

n = 4	3n = 12 = 7 + 5	**remainder 5**
n = 11	3n = 33 = 28 + 5	**remainder 5**
n = 18	3n = 54 = 49 + 5	**remainder 5**
Algebra: n = 7k + 4	3n = 21k + 12	**remainder 5**

IN THE LESSON

- Which numbers leave a remainder of four when divided by seven?
- Four itself, then eleven, eighteen, twenty five.
- Take the smallest: n equals four.
- Then three n is twelve.
- Divide twelve by seven: seven goes in once, with five left over.
- The remainder is five.
- Now check with a bigger one, to make sure this was not luck.
- Take n equal to eleven.
- Three n is thirty three.
- Seven times four is twenty eight, and thirty three minus twenty eight is five.
- The remainder is five again.
- One more, n equal to eighteen: three n is fifty four, and seven times seven is forty nine, so the remainder is five.
- Every example gives five.
- We could also write n as seven k plus four.
- Then three n is twenty one k plus twelve, and twenty one k is a multiple of seven, so only twelve matters, which leaves five.

11 Question 2: the traps in the choices

Multiply the remainder, then reduce it

A: 0	adds 3 to the 4 to make 7	**wrong operation**
B: 1	the quotient of 12 / 7	**quotient**
C: 4	assumes the remainder stays	**no change**
D: 5	12 reduced mod 7	**the answer**
E: 12	3 x 4, not reduced	**unreduced**

IN THE LESSON

- Now the traps.
- Choice A adds three to the remainder to get seven, instead of multiplying.
- Choice B divides twelve by seven and reports the quotient, one, rather than the remainder.
- Choice C assumes that multiplying n by three leaves the remainder unchanged.
- Choice E multiplies the remainder by three, getting twelve, but does not reduce it.
- A remainder must be smaller than the divisor, and twelve is larger than seven.
- That is the most common slip in remainder questions: forgetting to reduce.
- The answer is D, five.
- Notice how quickly the small number test did the work.
- Four, eleven and eighteen all agreed, and we had the answer in under a minute.

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12 Question 3: common divisors

- How many positive integers divide both 7,560 and 5,400?
- A. 16
- B. 31
- C. 32
- D. 48
- E. 64

IN THE LESSON

- Question three is a medium one.
- Read it with me.
- Do not start listing divisors: the two numbers are in the thousands, and a list would take far too long, and be easy to get wrong.
- How many positive integers divide both seven thousand five hundred sixty, and five thousand four hundred?
- Choice A.
- sixteen.
- Choice B.
- thirty one.
- Choice C.
- thirty two.
- Choice D.
- forty eight.
- Choice E.
- sixty four.
- Think prime factors, and the greatest common factor.

13 Question 3: prime factors

7,560 and 5,400 as products of primes

7,560	$2^3 \times 3^3 \times 5 \times 7$	**primes**
5,400	$2^3 \times 3^3 \times 5^2$	**primes**
Shared primes, smaller powers	$2^3 \times 3^3 \times 5$	**GCF**
GCF	8 x 27 x 5	**1,080**

IN THE LESSON

- Listing every divisor of numbers this size would take forever.
- Use primes instead.
- Seven thousand five hundred sixty is two cubed, times three cubed, times five, times seven.
- Check: eight times twenty seven is two hundred sixteen, times five is one thousand eighty, times seven is seven thousand five hundred sixty.
- Five thousand four hundred is two cubed, times three cubed, times five squared.
- Check: two hundred sixteen times twenty five is five thousand four hundred.
- Now the shared part: for each prime, take the smaller power.
- Two cubed.
- Three cubed.
- Five to the first power.
- Seven is only in the first number, so it is left out.
- The greatest common factor is two cubed times three cubed times five, which is one thousand eighty.

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14 Question 3: count the divisors

Divisors of 1,080 = $2^3 \times 3^3 \times 5$

Powers of 2: 0, 1, 2, 3	3 + 1	**4 choices**
Powers of 3: 0, 1, 2, 3	3 + 1	**4 choices**
Powers of 5: 0, 1	1 + 1	**2 choices**
Divisors of both numbers	4 x 4 x 2	**32**

IN THE LESSON

- A number divides both numbers exactly when it divides their greatest common factor.
- So the answer is the number of divisors of one thousand eighty.
- Use the exponent rule.
- Two cubed: exponent three, so four choices, using two zero, one, two or three times.
- Three cubed: four choices.
- Five to the first: two choices, using it or not.
- Multiply: four times four times two is thirty two.
- So thirty two positive integers divide both numbers.
- Compare with the traps.
- The number of divisors of the first number alone is four times four times two times two, which is sixty four.
- The number for the second is four times four times three, forty eight.
- The greatest common factor of sixty four and forty eight is sixteen, a trap that comes from a mix up.
- The answer is thirty two.

15 Question 3: the traps in the choices

Count the divisors of the GCF

A: 16	GCF of the counts 64 and 48	**mix-up**
B: 31	forgets the divisor 1	**off by one**
C: 32	4 x 4 x 2 divisors of the GCF	**the answer**
D: 48	divisors of 5,400 alone	**one number only**
E: 64	divisors of 7,560 alone	**one number only**

IN THE LESSON

- Now the traps.
- Choice A takes the greatest common factor of the two divisor counts, sixty four and forty eight, instead of counting shared divisors.
- Choice B counts the shared divisors but leaves out one, which divides everything.
- Choice D counts the divisors of five thousand four hundred alone, forgetting that the divisors of the other number must be shared too.
- Choice E counts the divisors of seven thousand five hundred sixty alone, so the seven is credited, even though it does not divide five thousand four hundred.
- The answer is C, thirty two.
- Notice how the prime factors turned a huge problem into reading three small numbers.
- This is the standard route for every common divisor question.

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16 Question 4: none of three

- How many integers from 1 to 300, inclusive, are divisible by none of 4, 6 and 10?
- A. 145
- B. 185
- C. 190
- D. 195
- E. 200

IN THE LESSON

- Question four is the hard one.
- Read it with me.
- Counting the numbers that fail every test is awkward, so count the numbers that pass at least one.
- How many integers from one to three hundred, inclusive, are divisible by none of four, six and ten?
- Choice A.
- one hundred forty five.
- Choice B.
- one hundred eighty five.
- Choice C.
- one hundred ninety.
- Choice D.
- one hundred ninety five.
- Choice E.
- two hundred.
- A numbers question with three overlapping conditions: a Venn diagram in disguise.

17 Question 4: count the multiples

Multiples from 1 to 300: single, then pairs

Multiples of 4, 6, 10	75, 50, 30	**155**
Both 4 and 6: multiples of 12	300 / 12	**25**
Both 4 and 10: multiples of 20	300 / 20	**15**
Both 6 and 10: multiples of 30	300 / 30	**10**
All three: multiples of 60	300 / 60	**5**

IN THE LESSON

- We want the numbers divisible by none of four, six and ten.
- It is easier to count the numbers divisible by at least one, and take them from three hundred.
- First count the multiples of each number up to three hundred.
- Multiples of four: three hundred divided by four is seventy five.
- Multiples of six: fifty.
- Multiples of ten: thirty.
- Adding gives one hundred fifty five, but numbers like twelve are counted twice, once as a multiple of four and once as a multiple of six.
- So we count the overlaps.
- A number divisible by both four and six is divisible by their least common multiple, twelve, and there are twenty five of those.
- Four and ten: twenty, fifteen of them.
- Six and ten: thirty, ten of them.

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18 Question 4: add, subtract, add back

At least one = singles - pairs + triple

Singles	75 + 50 + 30	**155**
Subtract pairs	25 + 15 + 10 = 50	**105**
Add back the triple	multiples of 60	**110**
Divisible by none	300 - 110	**190**

IN THE LESSON

- Now put it together, with inclusion and exclusion.
- Start with the singles, one hundred fifty five.
- Subtract the pairs, because each number divisible by two of them was counted twice: twenty five plus fifteen plus ten is fifty, and one hundred fifty five minus fifty is one hundred five.
- But a multiple of sixty, divisible by all three, was counted three times in the singles, and taken away three times in the pairs, so it has vanished.
- Add those back: five.
- One hundred five plus five is one hundred ten.
- So one hundred ten numbers are divisible by at least one.
- The numbers divisible by none are three hundred minus one hundred ten, which is one hundred ninety.
- As a check, the pattern is singles, minus pairs, plus triple.

19 Question 4: the traps in the choices

Singles, minus pairs, plus the triple

A: 145	subtracts only the singles	**no overlaps**
B: 185	adds the triple back twice	**double add**
C: 190	300 - 110	**the answer**
D: 195	forgets to add the triple	**missed triple**
E: 200	subtracts the triple	**wrong sign**

IN THE LESSON

- Now the traps, which all come from the last two steps.
- Choice A subtracts the three single counts, one hundred fifty five, from three hundred, without taking care of the overlaps.
- That gives one hundred forty five.
- Choice D subtracts the pairs but forgets to add back the multiples of sixty, which leaves one hundred five as divisible by at least one, and one hundred ninety five as the answer.
- Choice B adds the multiples of sixty back twice.
- Choice E subtracts them, when inclusion and exclusion adds them back.
- Each trap shows a different slip in the same sequence of signs.
- The answer is C, one hundred ninety.
- The sign pattern, plus, minus, plus, is the part to learn by heart for three overlapping groups.

20 Accurate and quick

- Key word: must, how many, greatest, least
- Small numbers first, then confirm by a rule
- Primes before factors, multiples, divisor counts
- Count so nothing is missed or doubled
- One-off choices: suspect an edge slip

IN THE LESSON

- Now habits that save time.
- First, read the question for its key word: must, how many, greatest, least.
- A word like must tells you to test several cases.
- Second, for odd and even and remainders, test small numbers first, and confirm with a rule.
- Third, write every number as primes before you look at factors, multiples or divisor counts.
- Fourth, when you count, count in a way that cannot miss: list in order, or use inclusion and exclusion, and always check the edges, such as whether one is included.
- Fifth, when a choice is one off from another, suspect an inclusion slip, such as the divisor one.
- And sixth, spend about two minutes, and remember that a small number test can usually eliminate three choices in thirty seconds.

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21 Quick check

1. Odd number plus twice any whole number?
   Answer: Odd: twice anything is even; odd + even is odd.
2. Remainder 2 on division by 5: remainder of double?
   Answer: 4: double 2 is 4, which is less than 5.
3. Divisors of 2 to the 4th times 3?
   Answer: (4 + 1)(1 + 1) = 10.

IN THE LESSON

- Now a quick check.
- Pause after each question and answer it yourself.
- Question one.
- Is the sum of an odd number and twice any whole number odd or even?
- The answer: odd, because twice anything is even, and odd plus even is odd.
- Question two.
- A number leaves a remainder of two when divided by five.
- What is the remainder when its double is divided by five?
- The answer: four, because the double of two is four, and four is less than five.
- Question three.
- How many divisors does a number have if it is two to the fourth power, times three?
- The answer: ten, because four plus one is five, times one plus one is two, and five times two is ten.

22 Recap: the number property checks

- Odd and even: small numbers, then the rules
- Remainders: write n out, then reduce
- Factors: primes, and the GCF for common divisors
- Divisor count: exponents plus one, multiplied
- Overlapping counts: singles - pairs + triple

IN THE LESSON

- Let us recap the technique.
- For odd and even questions, test small numbers, and confirm with the rules: odd plus even is odd, any product with an even factor is even, and twice anything is even.
- For remainder questions, write the number out, as a multiple of the divisor plus the remainder, test two or three examples, and reduce any remainder that is as large as the divisor.
- For factors, multiples and divisors, write every number as a product of primes.
- The common divisors are the divisors of the greatest common factor, and the divisor count is the product of the exponents plus one.
- For counts of numbers that meet a rule, use inclusion and exclusion with the signs plus, minus, plus.
- Check every answer against the edges of the question.

Summary notes for the Charta whiteboard lesson. Figures are from the lesson's made-up examples.

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