Charta | LESSON NOTES

Picking numbers or algebra: choose the right tool

Quantitative Reasoning

1 Two tools for one job

- Algebra: work with the letters; it always works
- Picking numbers: replace letters with simple numbers
- Arithmetic is harder to get wrong than algebra
- One pick can be unlucky: choices may agree at it

IN THE LESSON

- Imagine a question that asks which expression is equivalent to another one.
- You could expand and simplify with algebra, and that always works, but it is easy to slip on a sign.
- The other way is to choose a value for the letter, work out the target expression, work out each choice, and see which one matches.
- If x is two, everything becomes plain arithmetic.
- That is picking numbers.
- It turns a question about letters into a question about numbers, and numbers are harder to get wrong.
- The price is that one pick can be unlucky, because two different expressions can agree at one particular number.
- So the skill is choosing good numbers, and knowing when to check a second number, or when to use algebra.

2 Picking numbers: the steps

Pick, compute the target, test every choice

1. Does it hold for every value?	equivalent, in terms of	**signal**
2. Pick simple allowed numbers	obey every condition	**pick**
3. Work out the target	write the number down	**target**
4. Test each choice	same numbers, same order	**test**
5. Cross out and repeat	a new pick for survivors	**repeat**

IN THE LESSON

- Here are the steps.
- First, check that the question really holds for every value of the letters.
- Phrases like which of the following is equivalent, or in terms of x, are the signal.
- Second, pick simple numbers that obey every condition the question states, such as positive, or not equal to zero.
- Third, work out the value of the target expression with your numbers.
- Write it down.
- Fourth, work out the same value for each answer choice.
- Fifth, cross out every choice that does not match.
- If exactly one choice remains, you are done.
- If two or more remain, pick different numbers and test only the survivors.
- Most questions need no more than two picks, and you never need to start over.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

3 Good numbers and bad numbers

- Simple: small whole numbers
- Legal: obey every condition in the question
- Revealing: avoid 0 and 1, which hide differences
- Tailor made: 100 for percents, the ratio's own numbers

IN THE LESSON

- A good pick is simple, legal and revealing.
- Simple means small whole numbers like two, three or five, so the arithmetic is quick.
- Legal means obeying every condition in the question.
- If it says x is not zero, one or minus one, those values are banned, and picking one would make a fraction undefined.
- Revealing means the number does not make different choices equal.
- Zero and one are the worst offenders, because they hide differences: anything times zero is zero, and one raised to any power is one.
- Some numbers are tailor made.
- For a percentage problem, pick one hundred.
- For a ratio of two to three, pick two and three, or any multiple of them.
- For a question about odd and even, pick one example of each.

4 When algebra is better

- Facts that fix the numbers: algebra
- Find the one value of an unknown: algebra
- Patterns like a difference of squares: algebra
- Which expression is equivalent: pick numbers

IN THE LESSON

- Picking numbers has a limit.
- It works when the answer is the same for every legal value, or when the question asks which expression matches.
- It fails when the question gives facts that fix the numbers.
- If a question says that x squared minus y squared is forty eight, and x plus y is eight, then x and y are not free to be anything you like.
- Your picks would break the facts.
- And if a question asks you to find how many pupils a school has, there is one right number, and it comes from setting up and solving an equation.
- Algebra is also the quicker tool when a familiar pattern appears, such as the difference of two squares.
- A reader who knows the pattern can finish in seconds.

5 Choose the tool from the question

Can the letters be anything, or are they pinned down?

The question says	Best tool
Which is equivalent to...	Pick numbers
Ratio given; expression asked	Pick numbers that obey it
Find the value of an unknown	Algebra
Equations with a pattern	Algebra

IN THE LESSON

- Here is a quick table to choose the tool.
- If the question says which of the following is equivalent, or asks for an expression in terms of a letter, pick numbers.
- If it gives a ratio or a relationship and asks for an expression that depends only on that relationship, pick numbers that obey the relationship.
- If it asks for the value of an unknown from a set of facts, such as an equation, a system, or a story with a total, use algebra.
- And if you spot a pattern you know, such as a difference of squares, use algebra, because the pattern does the work for you.
- When you are unsure, spend ten seconds asking whether the letters can be anything, or whether they are pinned down.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

6 Question 1: equivalent expressions

- Which of the following is equivalent to (x + 5)(x - 3)?
- A. $x^2 + 2x - 15$
- B. $x^2 - 15$
- C. $x^2 - 2x - 15$
- D. $x^2 + 2x + 15$
- E. $x^2 + 8x - 15$

IN THE LESSON

- Question one is an easy one.
- Read it with me, and try it yourself first.
- Which of the following is equivalent to the product of x plus five, and x minus three?
- Choice A.
- x squared plus two x minus fifteen.
- Choice B.
- x squared minus fifteen.
- Choice C.
- x squared minus two x minus fifteen.
- Choice D.
- x squared plus two x plus fifteen.
- Choice E.
- x squared plus eight x minus fifteen.
- Notice the signal: which of the following is equivalent.
- Any value of x must work.

7 Question 1: a bad pick, x equals zero

Target: (x + 5)(x - 3). Pick x = 0

Target at x = 0	5 times -3	**-15**
A: $x^2 + 2x - 15$	all x terms vanish	**-15**
B: $x^2 - 15$	all x terms vanish	**-15**
C and E	also -15	**-15**
D: $x^2 + 2x + 15$	+15, not -15	**out**

IN THE LESSON

- Let us pick a number, and see how not to do it.
- Try x equals zero, the easiest number of all.
- The target is zero plus five, times zero minus three, which is five times minus three, minus fifteen.
- Now test the choices.
- Choice A, with x zero, is minus fifteen.
- It matches.
- Choice B is zero squared minus fifteen, which is also minus fifteen.
- It matches as well.
- Choice C is minus fifteen.
- Choice D is plus fifteen, so it is out.
- Choice E is minus fifteen as well.
- So zero rules out only one choice, and four are left.
- That is the problem with zero: it makes every term with an x vanish, so the choices look the same.
- Zero was legal, simple, and useless.
- We need a revealing number.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

8 Question 1: a good pick, x equals one

Target: (x + 5)(x - 3). Pick x = 1

Target at x = 1	6 times -2	**-12**
A: x² + 2x - 15	1 + 2 - 15	**-12 matches**
B: x² - 15, C: x² - 2x - 15	-14 and -16	**out**
D: x² + 2x + 15	18	**out**
E: x² + 8x - 15	-6	**out**

IN THE LESSON

- Now pick x equals one.
- The target is one plus five, times one minus three.
- Six times minus two is minus twelve.
- Test the choices with x equal to one.
- Choice A: one plus two minus fifteen is minus twelve.
- A match.
- Choice B: one minus fifteen is minus fourteen, not a match.
- Choice C: one minus two minus fifteen is minus sixteen.
- Choice D: one plus two plus fifteen is eighteen.
- Choice E: one plus eight minus fifteen is minus six.
- Only choice A matches.
- One pick, and four choices gone.
- As a safety check, try x equals two: the target is seven times minus one, which is minus seven, and choice A is four plus four minus fifteen, which is also minus seven.
- The answer is A.

9 Question 1: the traps in the choices

Wrong choices are built from algebra slips

A: x² + 2x - 15	matches at x = 1 and x = 2	**the answer**
B: x² - 15	drops the middle terms	**first and last**
C: x² - 2x - 15	middle sign wrong	**sign slip**
D: x² + 2x + 15	constant loses its sign	**sign slip**
E: x² + 8x - 15	adds 5 and 3	**wrong middle**

IN THE LESSON

- Look at what the wrong choices are.
- Choice B multiplies only the first terms and the last terms, dropping the middle terms.
- That is the classic slip of treating a product like a sum.
- Choice C has the wrong sign on the middle term.
- Choice D loses the negative sign on the constant.
- Choice E adds five and three for the middle coefficient, when it should combine five x and minus three x.
- Every one of these slips is natural in algebra, and none of them survives a single good pick.
- That is the advantage of picking numbers: the wrong choices are built from algebra mistakes, and arithmetic does not make those mistakes.
- The answer is A.
- And notice that x equals zero would not have separated A from B.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

10 Question 2: a ratio

- If x and y are positive numbers with x/y = 2/3, what is the value of $(3x^2 - y^2)/(xy)$?
- A. 9/2
- B. 23/6
- C. 3
- D. 1/2
- E. 1/6

IN THE LESSON

- Question two is a medium one.
- Read it with me.
- The two letters are linked by a ratio, and the expression has letters on the top and on the bottom.
- Try to guess whether it depends on the sizes of x and y, or only on the ratio between them.
- If x and y are positive numbers with x over y equal to two thirds, what is the value of three x squared minus y squared, all over x times y?
- Choice A.
- nine halves.
- Choice B.
- twenty three sixths.
- Choice C.
- three.
- Choice D.
- one half.
- Choice E.
- one sixth.
- Does the answer depend on the sizes of x and y?
- Or only on their ratio?

11 Question 2: pick numbers that obey the ratio

x / y = 2/3, so pick x = 2 and y = 3

Condition	2 / 3	**legal**
Top: $3x^2 - y^2$	3 x 4 - 9	**3**
Bottom: xy	2 x 3	**6**
Value	3 / 6	**1/2**
Second pair: x = 4, y = 6	12 / 24	**1/2**

IN THE LESSON

- The ratio is the only relationship, so the expression must depend only on it.
- Any numbers with x over y equal to two thirds will give the same answer.
- So pick the simplest: x is two and y is three.
- Check the condition: two over three.
- Legal.
- Now the expression.
- Three x squared is three times four, which is twelve.
- Y squared is nine.
- The top is twelve minus nine, which is three.
- The bottom is x times y, two times three, which is six.
- So the value is three over six, one half.
- That is choice D.
- Now test a second pair to be safe: x is four and y is six.
- The top is forty eight minus thirty six, twelve.
- The bottom is twenty four.
- Twelve over twenty four is one half again.
- The answer does not change.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

12 Question 2: the traps in the choices

Pick numbers that obey the ratio: the answer is one half

A: 9/2	reads $3x^2$ as $(3x)^2$	**misread power**
B: 23/6	swaps x and y	**ratio flipped**
C: 3	top only, forgets xy	**half the job**
D: 1/2	3 over 6	**the answer**
E: 1/6	divides by 18	**wrong bottom**

IN THE LESSON

- Now the traps.
- Choice A reads three x squared as three x, all squared.
- With x equal to two, that is thirty six on top, minus nine, twenty seven, over six, which is nine halves.
- A misread of the power.
- Choice B uses y over x equal to two thirds by mistake, swapping the letters.
- With x equal to three and y equal to two, the value is twenty three over six.
- Choice C computes only the top, three, and forgets to divide.
- Choice E divides three by eighteen, using three x y in place of x y.
- Every wrong choice is a natural slip, and the numbers show it at once.
- The answer is D, one half.
- Notice that picking numbers kept the work to one line of arithmetic.

13 Another place to pick: percent stories

Up 20%, then down 20%: pick a start of 100

Start (a pick)	any price will do	**100**
Up 20%	100 + 20	**120**
Down 20% of 120	120 - 24	**96**
Compared with the start	96 against 100	**4% lower**

IN THE LESSON

- Picking numbers is not only for letters.
- It also works in stories where a quantity is never given, such as a price that rises by twenty percent and then falls by twenty percent.
- How does the final price compare with the start?
- You could use algebra with a letter for the price, but it is easier to pick a start of one hundred, because every percentage of one hundred is just the percent number.
- Twenty percent more than one hundred is one hundred twenty.
- Twenty percent less than one hundred twenty is twenty four less, which is ninety six.
- The final price is ninety six against one hundred: four percent lower, and not unchanged, as many people guess.
- Because the start was a pick, the answer, four percent lower, holds for any starting price.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

14 Question 3: facts that pin it down

- If $x^2 - y^2 = 48$ and $x + y = 8$, what is the value of $x - y$?
- A. 1
- B. 6
- C. 7
- D. 40
- E. 56

IN THE LESSON

- Question three is an easy one, and a good one for algebra.
- Read it with me.
- Look at how the question is built: there are two facts, and both of them tie x and y together.
- Before any calculation, ask yourself whether you could pick numbers freely here.
- If x squared minus y squared equals forty eight, and x plus y equals eight, what is the value of x minus y?
- Choice A.
- one.
- Choice B.
- six.
- Choice C.
- seven.
- Choice D.
- forty.
- Choice E.
- fifty six.
- Ask first: can x and y be anything?
- Or are they pinned down?

15 Question 3: see the pattern

$x^2 - y^2 = 48$ and $x + y = 8$

Try x = 5, y = 3	sum 8, but 25 - 9 = 16	**breaks fact**
The pattern	$x^2 - y^2 = (x + y)(x - y)$	**factor**
Substitute	8 times (x - y) = 48	**equation**
Solve	x - y = 48 / 8	**6**
Check: x = 7, y = 1	49 - 1 = 48, 7 + 1 = 8	**true**

IN THE LESSON

- Can we pick numbers here?
- Not freely.
- The facts say x squared minus y squared is forty eight, and x plus y is eight.
- If you pick x as five and y as three, then x plus y is eight, but x squared minus y squared is sixteen, which breaks the first fact.
- The facts have pinned the numbers down.
- You would have to hunt for the right pair, which is slow.
- Algebra is better, and a famous pattern is hiding in plain sight.
- X squared minus y squared factors into x plus y, times x minus y.
- We know the product is forty eight, and we know x plus y is eight.
- So eight times x minus y is forty eight, and x minus y is forty eight divided by eight, which is six.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

16 Question 3: the traps in the choices

Algebra: factor, substitute, divide

A: 1	the value of y, not x - y	**wrong target**
B: 6	48 divided by 8	**the answer**
C: 7	the value of x alone	**wrong target**
D: 40	48 minus 8	**wrong operation**
E: 56	48 plus 8	**wrong operation**

IN THE LESSON

- Look at the traps.
- Choice A, one, is the value of y, and choice C, seven, is the value of x.
- Both are correct numbers for the situation, but the question asks for x minus y.
- After you find the numbers, always re-read the question.
- Choices D and E subtract and add the eight, instead of dividing the forty eight by it.
- They come from not knowing the factor pattern, which makes a guess at an operation.
- The check at the end settles it: x is seven and y is one, and seven squared minus one squared is forty eight, and seven plus one is eight.
- The answer is B.
- Notice that when the facts pin the numbers down, algebra was both faster and safer.

17 Question 4: a school

- At a school the ratio of teachers to pupils is 1 to 15. After 6 more teachers are hired and 40 pupils leave, the ratio is 1 to 10. How many pupils does the school have after the change?
- A. 100
- B. 300
- C. 340
- D. 80
- E. 260

IN THE LESSON

- Question four is a medium word problem.
- Read it with me.
- At a school the ratio of teachers to pupils is one to fifteen.
- After six more teachers are hired and forty pupils leave, the ratio is one to ten.
- How many pupils does the school have after the change?
- Choice A.
- one hundred.
- Choice B.
- three hundred.
- Choice C.
- three hundred forty.
- Choice D.
- eighty.
- Choice E.
- two hundred sixty.
- One right number is wanted, so think algebra.
- What is the unknown?

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

18 Question 4: set up the equation

t teachers and 15t pupils; then t + 6 and 15t - 40

New ratio 1 to 10	15t - 40 = 10(t + 6)	**equation**
Expand and solve	5t = 100	**t = 20**
Original pupils	15 x 20	**300**
Pupils after the change	300 - 40	**260**
Check: teachers 26	26 x 10	**260**

IN THE LESSON

- There is one right number, so use algebra, and let one letter stand for the unknown.
- Let t be the original number of teachers.
- The ratio of teachers to pupils is one to fifteen, so there are fifteen t pupils.
- After the change, there are t plus six teachers.
- And the pupils are fifteen t minus forty, because forty leave, not arrive.
- The new ratio is one to ten, so the pupils are ten times the teachers: fifteen t minus forty equals ten times, t plus six.
- Expand the right side: ten t plus sixty.
- Subtract ten t from both sides, and add forty: five t equals one hundred.
- So t is twenty.
- The question asks for the pupils after the change: fifteen times twenty is three hundred, minus forty, is two hundred sixty.

19 Question 4: the traps in the choices

Algebra: one unknown, one equation, then re-read the question

A: 100	adds 40 instead of subtracting	**wrong sign**
B: 300	pupils before the change	**wrong stage**
C: 340	adds 40 to the 300	**wrong sign**
D: 80	forgets the 6 new teachers	**missing step**
E: 260	300 - 40, ratio 26 to 260	**the answer**

IN THE LESSON

- Now the traps.
- Choice B, three hundred, is the original number of pupils.
- We found it on the way, but the question asks about after the change.
- Choice C adds forty to three hundred, when forty pupils leave.
- Choice A comes from the same sign slip, and choice D forgets the six new teachers when setting up the new ratio.
- The answer is E, two hundred sixty, and the check is neat: twenty six teachers and two hundred sixty pupils make a ratio of one to ten.
- Could picking numbers have solved this?
- You could try twenty teachers and test, but you would be guessing, since the story pins the numbers down.
- When a question asks for one value, set up the equation.

20 Accurate and quick

- Five seconds: free letters, or pinned down?
- Write the target value first; one choice per line
- Avoid 0 and 1; obey every condition
- Algebra: re-read what the question asks
- Stuck on one tool? Switch to the other

IN THE LESSON

- Now habits that make the choice fast.
- First, spend five seconds on the choice of tool: can the letters be anything, or are they pinned down?
- Second, when you pick numbers, write the target value first, and keep your work in a column, one choice below another.
- Third, avoid zero and one unless the question demands them, and never use a number that the question bans.
- Fourth, if two choices survive, pick again, and test only those two.
- Fifth, with algebra, write the unknown in words first, and after you solve, re-read the question to check what it asks.
- The answer to a word problem is often not the letter you solved for.
- Sixth, spend about two minutes a question, and if you are stuck on one tool, switch to the other.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.

21 Quick check

1. x = 0 leaves two choices matching. What now?
   Answer: Pick a new number, test only the two.
2. x / y = 3 / 4, expression asked: what to pick?
   Answer: x = 3 and y = 4, or any multiple.
3. Two equations, find x - y: which tool?
   Answer: Algebra: the facts pin the numbers down.

IN THE LESSON

- Now a quick check.
- Pause after each question and answer it yourself.
- Question one.
- A question asks which expression is equivalent to another, and you pick x equal to zero.
- Two choices match.
- What do you do?
- The answer: pick a different number, such as one or two, and test only the two survivors.
- Question two.
- A question says x over y is three over four, and asks for an expression in x and y.
- What do you pick?
- The answer: x equal to three and y equal to four, or any numbers with that ratio.
- Question three.
- A question gives two equations and asks for the value of x minus y.
- Tool?
- The answer: algebra, because the facts pin the numbers down.

22 Recap: picking numbers or algebra

- Free letters? Pick numbers. Pinned down? Algebra
- Simple, legal, revealing: avoid 0 and 1
- Target first, then every choice
- Algebra: name the unknown, solve, re-read
- Wrong choices are built from natural slips

IN THE LESSON

- Let us recap the technique.
- Ask whether the letters can be anything, or are pinned down.
- If the question asks which expression is equivalent, or for an expression that depends only on a stated relationship, pick numbers.
- Choose simple, legal, revealing numbers, and avoid zero and one.
- Work out the target first, then every choice, and cross out the mismatches.
- If two survive, pick again.
- If the question gives facts that pin the numbers down, or asks for the value of an unknown, use algebra.
- Look for patterns such as a difference of squares, name the unknown, set up the equation and solve it, and then re-read the question to see what it actually asks.
- The wrong choices are the slips you would make.

Summary notes for the Charta whiteboard lesson. Figures are from the lesson's made-up examples.

Charta Learning · charta-learning.com

© Charta Learning · charta-learning.com · For your own study. Please do not share or redistribute.