Charta | LESSON NOTES

Rates and work: one table, two ideas

Quantitative Reasoning

1 Amount equals rate times time

- Amount = rate x time
- Rate = amount / time; time = amount / rate
- Speed question: the amount is distance
- Work question: the whole job counts as 1

IN THE LESSON

- Here is the one formula behind every question in this lesson.
- Amount equals rate times time.
- If a car goes at sixty kilometres an hour for two hours, the distance is one hundred twenty.
- If a pipe fills a tank at a rate of one fifth of the tank every hour, then in three hours it fills three fifths.
- From the same formula you get the other two forms.
- Rate is amount divided by time.
- Time is amount divided by rate.
- The only thing that changes between a speed question and a work question is what the amount is: kilometres in one, and fractions of a job in the other.
- In a work question the whole job counts as one, so rates are fractions per hour.

2 The method: a table, then the totals

Rate, time, amount: fill the table

1. List the parts	legs, pipes, machines	**rows**
2. Fill a table	rate, time, amount	**table**
3. What is wanted?	total, average, time, rate	**goal**
4. Use totals	not averages of averages	**totals**
5. Check the units	hours against minutes	**convert last**

IN THE LESSON

- Here is the method.
- First, list the parts of the story, such as the legs of a journey, or the pipes and machines.
- Second, make a small table, with a row for each part and columns for rate, time and amount.
- Fill in whatever you know from the question, and work out each missing entry from the formula.
- Third, ask what the question wants: a total, an average, a time or a rate.
- Fourth, use totals, not averages of averages: total distance over total time for an average speed, and total work over combined rate for a time.
- Fifth, check the units.
- If rates are per hour and the answer is wanted in minutes, convert at the very end.
- Keep the table tidy, and the arithmetic will be short.

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3 Average speed is not the average of the speeds

- Average speed = total distance / total time
- Equal times: the simple average works
- Equal distances: it does not
- Rest and waiting count, if the question says so

IN THE LESSON

- The classic trap in speed questions is averaging the speeds.
- If you drive for an hour at thirty and an hour at ninety, you drive for equal times, and the average speed is sixty, the simple average.
- But if you drive thirty kilometres at thirty, and thirty kilometres at ninety, you spend a whole hour on the slow part and only twenty minutes on the fast part, so you are slower than sixty.
- The rule that always works is that the average speed is the total distance divided by the total time.
- Never add speeds and divide by the number of speeds, unless the times are equal.
- And remember that any time spent resting or waiting is part of the total time, if the question says including the rest.

4 Work and pipes: rates add

- Rate = 1 / the time to do the whole job
- Together: add the rates; a drain subtracts
- Time = work left / combined rate
- A delayed start: do the stretches separately

IN THE LESSON

- In a work question, find each worker's rate as a fraction of the job per hour.
- If a pipe fills a tank in six hours, its rate is one sixth of the tank each hour.
- When two pipes work together, their rates add.
- One sixth plus one tenth per hour is the combined rate.
- A drain works against you, so its rate is subtracted.
- To find a time, divide the amount of work left by the combined rate.
- If the tank is already one third full, only two thirds is left to do.
- And if a machine works alone for a while before another joins, work out the first stretch separately, then the second.
- Always ask how much of the job remains, and which workers are on at that stage.

5 Rate questions: which first move?

Name the parts, then total

The question	First move
Legs of a journey	table of distance and time
Average speed	total distance / total time
Workers or pipes together	add the rates (1 / time)
Late start or part done	split into stretches

IN THE LESSON

- Here is a quick guide to the first move.
- If the question gives distances and speeds for several legs, make a table of legs and total the distance and the time.
- If it asks for an average speed, find the total distance and the total time, and divide.
- If it gives times for each worker to do a job alone, convert each to a rate, one over the time, and add.
- If part of the job is already done, subtract it from one before you divide.
- If a worker joins late or leaves early, split the story into stretches, each with its own set of workers.
- And in every question, decide on the unit of time at the start, and convert the result at the end.

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6 Question 1: a cyclist and a lake

- A cyclist rides 36 kilometres to a lake at 12 kilometres per hour, rests there for 2 hours, and rides back along the same route at 9 kilometres per hour. What is her average speed, in kilometres per hour, for the whole trip including the rest?
- A. 4
- B. 8
- C. 10 2/7
- D. 10.5
- E. 12

IN THE LESSON

- Question one is an easy one.
- Read it with me, and try it yourself first.
- Notice the word including, near the end.
- A cyclist rides thirty six kilometres to a lake at twelve kilometres per hour, rests there for two hours, and rides back along the same route at nine kilometres per hour.
- What is her average speed, in kilometres per hour, for the whole trip including the rest?
- Choice A.
- four.
- Choice B.
- eight.
- Choice C.
- ten and two sevenths.
- Choice D.
- ten point five.
- Choice E.
- twelve.
- We want the average speed for the whole trip including the rest.

7 Question 1: the table of legs

Legs of the trip: distance, time

Ride out	36 / 12	**3 hours**
Rest	2 hours, no distance	**2 hours**
Ride back	36 / 9	**4 hours**
Total distance, total time	72 km, 3 + 2 + 4	**9 hours**
Average speed	72 / 9	**8 km/h**

IN THE LESSON

- Make a table of the three parts of the trip: the ride out, the rest, and the ride back.
- Ride out: thirty six kilometres at twelve kilometres an hour takes thirty six over twelve, which is three hours.
- The rest takes two hours, with no distance.
- Ride back: thirty six kilometres at nine an hour takes thirty six over nine, which is four hours.
- The total time is three plus two plus four, which is nine hours.
- The total distance is thirty six out and thirty six back, which is seventy two kilometres.
- Now the average speed is the total distance over the total time: seventy two divided by nine, which is eight kilometres per hour.
- The rest counts, because the question says the whole trip including the rest.
- The answer is eight.

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8 Question 1: the traps in the choices

Total distance over total time

A: 4	36 over 9: one way only	**half the distance**
B: 8	72 km in 9 hours	**the answer**
C: 10 2/7	72 over 7: rest left out	**no rest**
D: 10.5	averages 12 and 9	**average of speeds**
E: 12	takes the outward speed	**first speed**

IN THE LESSON

- Look at the traps.
- Choice A divides the one way distance, thirty six, by the total time, instead of the round trip distance, seventy two.
- Choice C leaves the two hour rest out of the total time, so it divides seventy two by seven.
- The question said including the rest, so that is not allowed.
- Choice D averages the two riding speeds, twelve and nine, to get ten and a half, which is the classic mistake.
- Choice E takes the outward speed as the average.
- The answer is B, eight kilometres per hour.
- Notice how the table made every one of these slips visible.
- The total distance and the total time are two numbers, and the average speed is just their quotient.

9 Question 2: pipes and a drain

- A tank is one-third full. Pipe A alone could fill an empty tank in 6 hours, pipe B alone in 10 hours, and a drain C could empty a full tank in 15 hours. If A, B and C are all opened together, how many minutes will it take to fill the tank?
- A. 100
- B. 120
- C. 150
- D. 200
- E. 300

IN THE LESSON

- Question two is a medium one.
- Read it with me.
- Notice that the tank is already part full, and that one of the three is a drain.
- A tank is one third full.
- Pipe A alone could fill an empty tank in six hours, pipe B alone in ten hours, and a drain C could empty a full tank in fifteen hours.
- If A, B and C are all opened together, how many minutes will it take to fill the tank?
- Choice A.
- one hundred.
- Choice B.
- one hundred twenty.
- Choice C.
- one hundred fifty.
- Choice D.
- two hundred.
- Choice E.
- three hundred.
- The unit of the answer is minutes, but the rates are in hours.

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10 Question 2: the combined rate

Rates in tanks per hour; the drain is negative

Pipe A	1 / 6 = 5/30	**+ 5/30**
Pipe B	1 / 10 = 3/30	**+ 3/30**
Drain C	-1 / 15 = -2/30	**- 2/30**
All three open	(5 + 3 - 2) / 30	**1/5 per hour**

IN THE LESSON

- First the rates, as fractions of a full tank per hour.
- Pipe A fills the tank in six hours, so its rate is one sixth.
- Pipe B takes ten hours, so one tenth.
- The drain empties a full tank in fifteen hours, so it works at minus one fifteenth.
- To add fractions, use a common denominator of thirty.
- One sixth is five thirtieths.
- One tenth is three thirtieths.
- One fifteenth is two thirtieths.
- The combined rate is five plus three minus two, which is six thirtieths, which is one fifth.
- So with all three open, the tank fills at one fifth of a tank every hour.
- The drain slows the filling, but the pipes win.
- Now we know how fast the tank fills.
- Next, how much is left to fill?

11 Question 2: the work left, then the time

Work left = 1 - 1/3 = 2/3 of the tank

Work left	1 - 1/3	**2/3 of a tank**
Time in hours	2/3 divided by 1/5	**10/3 hours**
Convert to minutes	10/3 x 60	**200**
Sense check	whole tank: 5 hours, two thirds	**3 1/3 hours**

IN THE LESSON

- The tank is already one third full, so two thirds of a tank remain to be filled.
- The time is the work left divided by the rate: two thirds divided by one fifth.
- Dividing by a fraction is multiplying by its reciprocal, so two thirds times five is ten thirds of an hour.
- Ten thirds of an hour is three and a third hours.
- The question asks for minutes, so convert at the end: ten thirds times sixty minutes is two hundred minutes.
- The answer is two hundred.
- Check the unit: three and a third hours is three hours twenty minutes, which is two hundred minutes.
- One more check that it is sensible: at one fifth of a tank an hour, filling the whole tank would take five hours, and two thirds of that is three and a third.

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12 Question 2: the traps in the choices

Sign of the drain, and the work left

A: 100	fills the third already full	**wrong amount**
B: 120	adds the drain as an inlet	**drain sign**
C: 150	leaves out the drain	**no drain**
D: 200	2/3 divided by 1/5, in minutes	**the answer**
E: 300	fills the whole tank	**ignores the third**

IN THE LESSON

- Now the traps.
- Choice A uses the one third that is already full as the amount still to fill, and gets one hundred minutes.
- Choice B adds the drain's rate as though it were a third inlet, so the combined rate is one third and the time is two hours.
- Choice C leaves out the drain, so the pipes alone fill at four fifteenths.
- Choice E fills the whole tank from empty, and forgets that one third is already full, five hours.
- Each wrong choice is a different way of mishandling one of three facts: the drain, the starting level, or the amount left.
- The answer is D, two hundred minutes.
- Writing the rates with their signs, and the work left, is what protects you.

13 Question 3: a delayed start

- Machine A packs 40 boxes an hour and machine B packs 60 boxes an hour. An order of 1,060 boxes is started by A alone, and B joins in after 90 minutes. How many hours after A starts is the order complete?
- A. 10
- B. 10.6
- C. 11.5
- D. 12.1
- E. 26.5

IN THE LESSON

- Question three is a medium one.
- Read it with me.
- One machine starts alone, and the other joins later.
- Machine A packs forty boxes an hour and machine B packs sixty boxes an hour.
- An order of one thousand sixty boxes is started by A alone, and B joins in after ninety minutes.
- How many hours after A starts is the order complete?
- Choice A.
- ten.
- Choice B.
- ten point six.
- Choice C.
- eleven point five.
- Choice D.
- twelve point one.
- Choice E.
- twenty six point five.
- Split the story into two stretches.

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14 Question 3: the head start

Stretch 1: A alone. Stretch 2: A and B together

Stretch 1: A alone	90 minutes = 1.5 hours	**1.5 hours**
Boxes packed in stretch 1	40 x 1.5	**60 boxes**
Boxes left	1,060 - 60	**1,000**
Stretch 2: combined rate	40 + 60	**100 an hour**
Time for stretch 2	1,000 / 100	**10 hours**

IN THE LESSON

- Split the story into two stretches.
- In the first stretch, only machine A works, for ninety minutes.
- Ninety minutes is one and a half hours, so convert the units straight away, because the rates are per hour.
- A packs forty boxes an hour, so in one and a half hours it packs sixty boxes.
- That is the head start.
- The order has one thousand sixty boxes, so after the first stretch one thousand boxes remain.
- In the second stretch, both machines work.
- Their rates add: forty plus sixty is one hundred boxes an hour.
- One thousand boxes at one hundred boxes an hour takes ten hours.
- So the second stretch lasts ten hours.
- We are not finished, because the question asks for the time after A starts.

15 Question 3: the traps in the choices

Add the stretches to get the total time

A: 10	only the time after B joins	**forgets the 1.5**
B: 10.6	both machines from the start	**ignores the delay**
C: 11.5	1.5 + 10 hours	**the answer**
D: 12.1	adds the 1.5 hours twice	**double count**
E: 26.5	A alone does the whole order	**ignores B**

IN THE LESSON

- The question asks how many hours after A starts, so add the two stretches: one and a half hours, plus ten hours, is eleven and a half hours.
- The answer is C.
- Now the traps.
- Choice A gives only the second stretch, ten hours, and leaves out A's first ninety minutes.
- Choice B has both machines working from the very start, so one thousand sixty over one hundred, ignoring B's delay.
- Choice D takes the whole order at the combined rate, and then adds the ninety minutes, so A's head start is counted twice.
- Choice E lets machine A pack the whole order alone.
- Each trap is a way of mishandling the head start, and the cure is to do the stretches separately, with the work left between them.

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16 Question 4: average speed of a journey

- A driver covers the first half of a journey's distance at 30 kilometres per hour. For the second half of the distance she drives for equal times at 30 and at 90 kilometres per hour. What is her average speed, in kilometres per hour, for the whole journey?
- A. 30
- B. 40
- C. 45
- D. 50
- E. 60

IN THE LESSON

- Question four is the hard one.
- Read it with me.
- The second half of the distance is driven in a special way.
- A driver covers the first half of a journey's distance at thirty kilometres per hour.
- For the second half of the distance she drives for equal times at thirty and at ninety kilometres per hour.
- What is her average speed, in kilometres per hour, for the whole journey?
- Choice A.
- thirty.
- Choice B.
- forty.
- Choice C.
- forty five.
- Choice D.
- fifty.
- Choice E.
- sixty.
- Equal distances for the two halves, but equal times within the second half.

17 Question 4: the second half first

Each half of the distance is d; equal times in the second half

Equal times t at 30 and at 90	distance 30t + 90t	**120t**
Time for the second half	t + t	**2t**
Average, second half	120t / 2t	**60 km/h**
Careful	simple average: equal times	**rule**

IN THE LESSON

- Let the distance of each half be d.
- We do not know d, and we will see that it cancels.
- Look at the second half, which is driven for equal times at thirty and at ninety.
- When the times are equal, the average speed is the simple average of the two speeds.
- That is the one case where the simple average works: thirty plus ninety is one hundred twenty, divided by two is sixty.
- Let us check that with distances.
- Say each part lasts t hours.
- The distance is thirty t plus ninety t, which is one hundred twenty t.
- The time is two t.
- So the average is one hundred twenty t over two t, which is sixty.
- So the second half of the distance is driven at an average of sixty kilometres per hour.

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18 Question 4: the whole journey

Equal distances: total distance over total time

First half	d / 30	**d/30 hours**
Second half	d / 60	**d/60 hours**
Total time	d/30 + d/60	**d/20 hours**
Average speed	2d divided by d/20	**40 km/h**

IN THE LESSON

- Now the whole journey, which is two halves of equal distance.
- The first half, a distance d at thirty, takes d over thirty hours.
- The second half, a distance d at an average of sixty, takes d over sixty hours.
- The total time is d over thirty plus d over sixty.
- With a common denominator of sixty, that is two d over sixty plus d over sixty, which is three d over sixty, which is d over twenty.
- The total distance is two d.
- So the average speed is two d divided by d over twenty, which is two times twenty, forty kilometres per hour.
- This time the speeds were averaged over equal distances, and the answer, forty, is lower than the simple average of thirty and sixty, which is forty five.
- Slower speeds take more time, so they count more.

19 Question 4: the traps in the choices

Equal times: simple average. Equal distances: total over total

A: 30	assumes the speed stays at 30	**no change**
B: 40	2d over d/20	**the answer**
C: 45	averages 30 and 60	**average of speeds**
D: 50	averages 30, 30 and 90	**three speeds**
E: 60	the second half only	**half the journey**

IN THE LESSON

- Now the traps.
- Choice A takes the average speed to stay at thirty, as though the second half could not change it.
- Choice C averages the speeds of the two halves, thirty and sixty, instead of dividing the total distance by the total time.
- This is the classic trap, and notice that it is correct only for equal times.
- Choice D averages the three speeds, thirty, thirty and ninety.
- Choice E gives the average speed for the second half only.
- The answer is B, forty.
- The whole lesson in one line: use the simple average only when the times are equal.
- When the distances are equal, divide the total distance by the total time, and the slower speed counts for more.

20 Accurate and quick

- A table: rate, time, amount for each part
- Convert minutes to hours at the start
- Average speed: total distance over total time
- Work: rates as fractions; drains are negative
- Sense check against the slowest and fastest

IN THE LESSON

- Now habits that keep rate questions quick and safe.
- First, write the parts of the story in a column, and fill in a rate, a time and an amount for each.
- Second, decide on the unit of time at the start, and convert any minutes to hours immediately.
- Third, for average speeds, always say total distance over total time, aloud or on paper.
- Fourth, for work, write each rate as a fraction, one over the time, and give the drains a minus sign.
- Fifth, ask how much work is left before you divide.
- Sixth, sense check the answer.
- Is the time longer than the slowest worker alone would take?
- Is the average speed between the slowest and fastest speeds?
- Seventh, spend about two minutes, and trust the table over your instinct.

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21 Quick check

1. 60 km at 60, then 60 km at 30: average speed?
   Answer: 40: 120 km in 3 hours.
2. Pipes of 4 and 12 hours together?
   Answer: 3 hours: 1/4 + 1/12 = 1/3 per hour.
3. A 6-hour job, a third done: time left?
   Answer: 4 hours: two thirds of 6.

IN THE LESSON

- Now a quick check.
- Pause after each question and answer it yourself.
- Question one.
- A car drives sixty kilometres at sixty kilometres an hour, and then sixty kilometres at thirty.
- What is the average speed?
- The answer: forty, because the total distance is one hundred twenty, and the total time is one plus two, which is three hours.
- Question two.
- A pipe fills a tank in four hours and another in twelve hours.
- How long will they take together?
- The answer: three hours, because one quarter plus one twelfth is one third per hour.
- Question three.
- A job takes a worker six hours alone, and she has done a third.
- How long to finish?
- The answer: four hours, since two thirds of six is four.

22 Recap: rates and work

- Amount = rate x time, in a table
- Average speed: total distance / total time
- Work: rates add; a drain subtracts
- Work left / combined rate = time
- Delayed start: stretches, then add the times

IN THE LESSON

- Let us recap the technique.
- Everything rests on amount equals rate times time.
- Make a table with a row for each part of the story, and fill in rate, time and amount.
- For average speed, divide the total distance by the total time, and include any rest if the question says so.
- The simple average of speeds is correct only for equal times.
- For work and pipes, write each rate as a fraction of the job per hour, add the rates of workers who are on together, and subtract a drain.
- Find the work left, and divide it by the combined rate.
- For a delayed start, do the stretches separately, and add the times.
- Keep the units straight, convert at the end, and check that the answer is sensible.

Summary notes for the Charta whiteboard lesson. Figures are from the lesson's made-up examples.

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