Duration: modified, Macaulay, money duration and PVBP

Fixed Income

1 A story: the seesaw

- Yields up: bond prices down
- Yields down: bond prices up
- How far does the price move?
- Duration measures that sensitivity

IN THE LESSON

- Think of a seesaw in a playground.
- Push down one end and the other end goes up.
- Bond prices and market yields sit on a seesaw like that.
- The yield is the return a buyer earns if they pay today's price and hold the bond to the end.
- When yields in the market rise, the price of a bond you already own goes down.
- When yields fall, its price goes up.
- Every bond investor knows that.
- The question that matters is how far the price moves.
- A bond that barely moves is calm.
- A bond that swings is risky.
- Duration is the number that answers that question, and we start with the one used most, modified duration.

2 Meet Dunmore Utilities

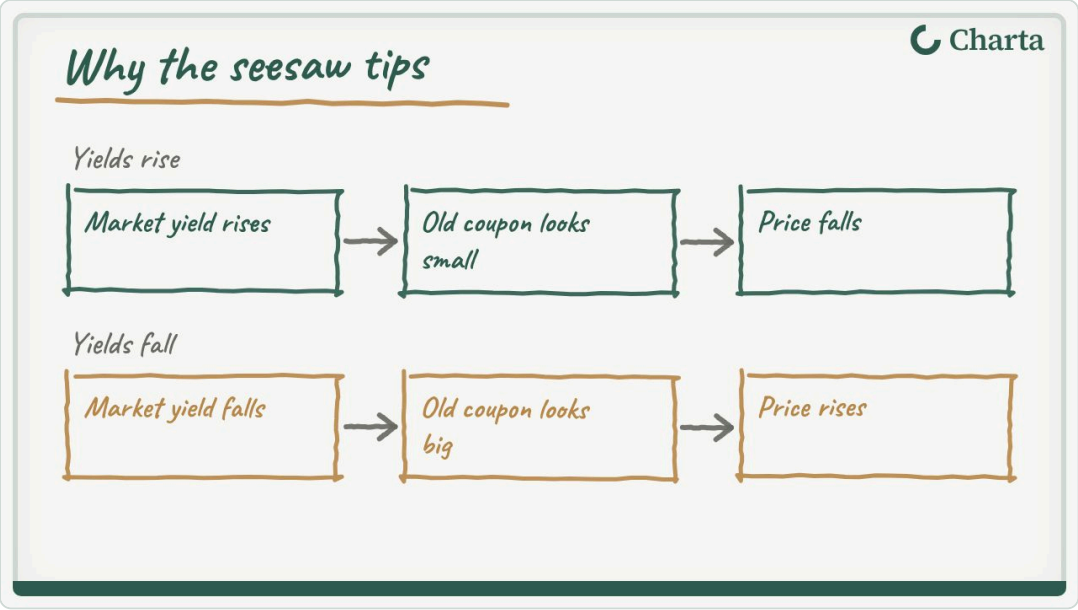
Dunmore Utilities, Bond B (made-up example)

Bond B	Value
Face value	100
Coupon, paid twice a year	5%
Life	10 years
Yield today, so price 100	5%

IN THE LESSON

- To make this concrete we follow one company, a power and water firm called Dunmore Utilities.
- Every figure is a made-up round number.
- Dunmore has a bond we will call Bond B.
- Its face value is one hundred.
- It pays a coupon, which is fixed interest, of five per cent a year, split into two payments, one every six months.
- It has ten years to run.
- We say prices per one hundred of face value.
- Today investors demand a yield of five per cent.
- When the coupon and the yield match like this, the bond sells exactly at its face value, so the price is one hundred.
- We will use this bond again and again.

3 Why the seesaw tips



IN THE LESSON

- Why do prices fall when yields rise?
- Picture the market after yields have risen.
- A new bond now pays a higher rate than Bond B's fixed coupon.
- Nobody wants to pay full price for the older, lower coupon when a better one is on offer.
- So Bond B's price has to fall until its return matches what the market now offers.
- That chain is the whole story.
- The coupon never changes.
- What changes is the rate used to value the promised payments, and a higher rate makes those payments worth less today.
- Falling yields run the same chain in reverse.
- Duration puts a number on how strong the effect is.

4 Estimating the price change

%ΔPrice ≈ −ModDur × ΔYield

Price change, %	Estimated, in per cent
Modified duration	Given, from the bond's terms
Yield change	In percentage points

IN THE LESSON

- Here is the rule.
- Modified duration is the approximate percentage change in a bond's price for a change in yield of one percentage point.
- So to estimate a price change, multiply modified duration by the change in yield, and put a minus sign in front, because the seesaw works the other way round.
- A modified duration of seven means about seven per cent lost for each one percentage point rise in yield.
- Because duration is a slope, it works well only for small moves.
- The size of a move is usually quoted in basis points.
- One basis point is one hundredth of one per cent, so fifty basis points is one half of one per cent.

5 Example 1: yields rise half a point

Bond B, modified duration 7.79. Yield rises 50 bp (0.50%)

Yield change		**+0.50%**
Estimated price change	−7.79 × 0.50%	**-3.9%**
Estimated new price	100 × (1 − 3.9%)	**96.1**
Exact price at 5.50%	repriced in full	**96.19**

IN THE LESSON

- Example one.
- Bond B has a modified duration of seven point seven nine.
- We will see where that comes from shortly.
- Suppose market yields rise by fifty basis points, which is half a percentage point.
- The estimate is minus seven point seven nine times zero point five, which is minus three point nine per cent.
- A fall of three point nine per cent takes the price from one hundred to about ninety-six point one.
- Now check against the truth.
- Repricing the bond properly at a yield of five and a half per cent gives ninety-six point one nine, a fall of three point eight one per cent.
- The estimate is very close, because the move was small.

6 Example 2: yields fall a quarter point

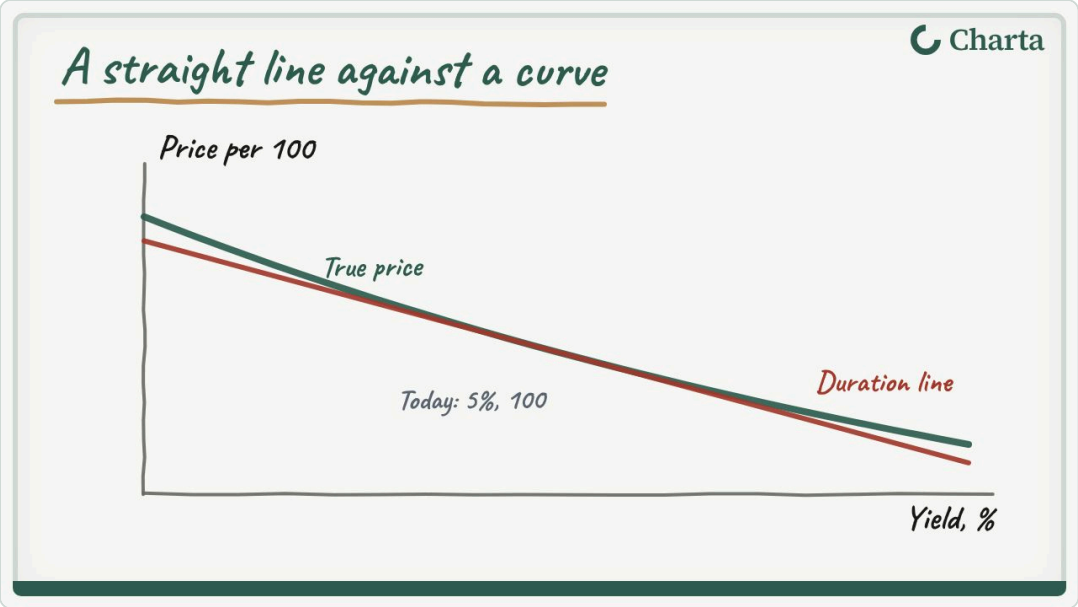
Bond B, modified duration 7.79. Yield falls 25 bp (0.25%)

Yield change	25 bp = 0.25%, not 25	**−0.25%**
Estimated price change	+7.79 × 0.25%	**+1.95%**
Estimated new price	100 + 1.95	**101.95**
Exact price at 4.75%	repriced in full	**101.97**

IN THE LESSON

- Example two, in the other direction, and with the unit trap that costs marks.
- Yields fall by twenty-five basis points.
- That is zero point two five of a percentage point, not twenty-five.
- The estimate is seven point seven nine times zero point two five, which is a gain of one point nine five per cent.
- The price rises from one hundred to about one hundred and one point nine five.
- Repricing the bond at four and three quarters per cent gives one hundred and one point nine seven, a gain of one point nine seven per cent.
- Again the estimate is close.
- Notice that this time the true gain is a touch larger than the estimate.
- Keep that in mind, because the next two slides explain it.

7 A straight line against a curve



IN THE LESSON

- Here is the same idea as a picture.
- The bottom axis is the market yield.
- The side axis is the price of Bond B per one hundred of face value.
- The curve is the true price at each yield.
- It slopes down, because higher yields mean lower prices, and it bends, so it is not a straight line.
- The straight line is the duration estimate.
- It touches the curve at today's point, a yield of five per cent and a price of one hundred.
- Away from that point the line drifts below the curve on both sides.
- That means duration understates a gain and overstates a loss.
- The gap is called convexity, and it is small for small moves.

8 Example 3: a very big move

Bond B, modified duration 7.79. Yield moves 300 bp (3.00%)

Duration estimate, either way	7.79 × 3.00%	**±23.37%**
Yield up to 8%: exact price		**79.61**
true change	milder than estimated	**-20.39%**
Yield down to 2%: exact price		**127.07**
true change	bigger than estimated	**+27.07%**

IN THE LESSON

- Example three shows the limit.
- Now yields move by three whole percentage points, or three hundred basis points.
- The duration estimate is seven point seven nine times three, which is twenty-three point three seven per cent, in each direction.
- The truth is different.
- If the yield rises to eight per cent, the exact price is seventy-nine point six one, a fall of twenty point three nine per cent.
- That is less than the estimate.
- If the yield falls to two per cent, the exact price is one hundred and twenty-seven point zero seven, a gain of twenty-seven point zero seven per cent, more than the estimate.
- So for large moves duration is rough, and the error always favours the holder of an ordinary bond.

9 A story: the parcels

- Macaulay duration: weighted average time to the cash
- Weight = present value of a payment ÷ price
- It is measured in years
- It is the starting point for the other two

IN THE LESSON

- Now we ask where the modified duration came from.
- Imagine a courier delivering parcels to you over the next few years, a small one every six months and a big one at the very end.
- Ask: on average, how long do I wait for my parcels?
- You would not just average the dates.
- The big parcel matters more, so you give it more weight.
- That weighted average waiting time is exactly Macaulay duration, named after the man who first used it.
- For a bond, the parcels are the coupons and the final repayment, and the weight of each is the share of the bond's price that it is worth today.
- The answer is a number of years.

10 Macaulay duration

MacDur = Σ [t × (PV of flow t ÷ Price)]

Time, t (years)	When the payment arrives
Weight of flow t	PV of that payment ÷ price
Price	Sum of all the PVs

IN THE LESSON

- Here is the recipe.
- For every payment the bond makes, take its present value, which is what it is worth today after discounting at the yield.
- Divide by the bond's price to get its weight.
- The weights add up to one, because the present values add up to the price.
- Then multiply each weight by the time of that payment, in years, and add everything up.
- That total is the Macaulay duration.
- One useful fact: a bond that pays nothing until the end, called a zero-coupon bond, has only one payment, with a weight of one, so its Macaulay duration is exactly its time to maturity.

11 Example 4: Macaulay duration by hand

Bond A: 2 years, 6% coupon twice a year, yield 5%

Year 0.5: pays 3, PV 2.93	weight 2.9%	**0.014**
Year 1.0: pays 3, PV 2.86	weight 2.8%	**0.028**
Year 1.5: pays 3, PV 2.79	weight 2.7%	**0.041**
Year 2.0: pays 103, PV 93.31	weight 91.6%	**1.832**

IN THE LESSON

- Example four, by hand, on a small bond so the sums fit.
- Dunmore's Bond A lasts two years, pays a six per cent coupon in two payments a year, and has a yield of five per cent.
- So it pays three, three, three, and then one hundred and three.
- Each half year we divide by one point zero two five, the half year yield, once for each period.
- The present values are two point nine three, two point eight six, two point seven nine and ninety-three point three one.
- The weights are two point nine, two point eight, two point seven and ninety-one point six per cent.
- Multiply each by its time in years.
- The last column is what we add.

12 From Macaulay to modified

Bond A, from the last slide

Price: sum of the PVs	2.93 + 2.86 + 2.79 + 93.31	**101.88**
Macaulay duration	sum of the last column	**1.92 years**
Modified duration	1.92 ÷ (1 + 0.05/2)	**1.87**
Meaning	yield up 1%: price falls about	**1.87%**

IN THE LESSON

- Now add up.
- The present values sum to the price of Bond A, one hundred and one point eight eight.
- The last column sums to one point nine two years.
- That is the Macaulay duration: on average, the cash arrives in just under two years, which makes sense, because most of the value is in the final payment.
- Now the step to modified duration.
- Divide Macaulay duration by one plus the yield per period.
- The yield is five per cent a year, paid in two periods, so the yield per period is two and a half per cent, and we divide by one point zero two five.
- The answer is one point eight seven.
- Modified duration is always a bit smaller than Macaulay duration.

13 Modified duration

ModDur = MacDur ÷ (1 + y/m)

Macaulay duration	A weighted time, in years
Yield per year, y	The annual yield, as a decimal
Periods a year, m	Payments a year: two if semiannual

IN THE LESSON

- Here is the link written out.
- Modified duration equals Macaulay duration divided by one plus the yield per period.
- The yield per period is the annual yield divided by the number of payments a year.
- This is where questions hide a trap.
- For a bond paying twice a year, divide by one plus half the annual yield, not by one plus the whole annual yield.
- For a bond paying once a year, the periods match.
- Remember the units, too.
- Macaulay duration is a time, in years.
- Modified duration is a sensitivity, the percentage price change for a one percentage point change in yield.
- They are close in size, but they say different things.

14 Money duration

MoneyDur = ModDur × Full price of the position

Money duration	In currency, per 1% yield move
Modified duration	From the last slide
Full price of position	Market value plus accrued

IN THE LESSON

- Percentages hide how much money is at stake.
- A hundred dollar bond and a hundred million dollar holding can share the same modified duration, yet one loses far more cash.
- Money duration fixes that.
- Multiply modified duration by the full price of the position, that is, its market value, including accrued interest, which is interest earned since the last coupon and not yet paid.
- The result is in currency.
- It says how much the value changes, in money, for a one percentage point change in yield.
- Some books call it dollar duration.
- Money duration is the bridge between a percentage measure and a trading decision.

15 Example 5: the same percentage, different money

Positions in $ millions: Bond A 20m par, Bond B 10m par

Bond A position value	20 × 101.88 ÷ 100	**20.38**
Bond A money duration	1.87 × 20.38	**38.11**
loss if yield +1%	38.11 × 1%	**−0.381**
Bond B money duration	7.79 × 10	**77.9**
loss if yield +1%	77.9 × 1%	**−0.779**

IN THE LESSON

- Example five.
- Dunmore holds twenty million of face value of Bond A, whose price is one hundred and one point eight eight, so the position is worth twenty point three eight million.
- Money duration is one point eight seven times twenty point three eight, about thirty-eight point one one million.
- So if yields rise by one percentage point, the position loses roughly zero point three eight one million.
- A rise of twenty-five basis points costs about zero point zero nine five million.
- Now compare ten million of Bond B, worth ten million.
- Its modified duration is seven point seven nine, so its money duration is about seventy-seven point nine million, and one percentage point costs about zero point seven seven nine million.
- Bond B is far riskier, yet its position is only half the size.

16 Three measures side by side

What each one measures and its unit

Measure	Unit
Macaulay duration	years
Modified duration	% per 1%
Money duration	$ per 1%
PVBP	$ per 1 bp

IN THE LESSON

- Here are the three measures together.
- Macaulay duration is a weighted average time, measured in years.
- It tells you when, on average, the cash arrives.
- Modified duration is the percentage change in price for a one percentage point change in yield.
- It tells you how sensitive the price is, and it is what we use in the estimate.
- Money duration is modified duration times the full price.
- It tells you how much money the position gains or loses for that one point.
- Later we will meet a fourth, the price value of a basis point, which is money duration for one basis point.
- Keep the units straight and most questions become easy.

17 Mistakes to avoid so far

- 1. 25 bp = 0.25%, not 25%
- 2. Yield up means price down: keep the minus sign
- 3. Macaulay is years; modified is a % sensitivity
- 4. Semiannual: divide by (1 + y/2), not (1 + y)
- 5. The estimate is only good for small moves

IN THE LESSON

- Before we go on, here are mistakes that cost marks.
- One.
- Mixing up the units of a yield change.
- Twenty-five basis points is zero point two five per cent, not twenty-five.
- Two.
- Dropping the minus sign.
- When yields rise, the price falls, and the estimate must be negative.
- Three.
- Treating Macaulay duration as a percentage.
- It is a time in years, and modified duration is smaller after dividing.
- Four.
- Dividing by one plus the full annual yield for a semiannual bond.
- Use one plus half the yield.
- Five.
- Trusting the estimate for a huge move.
- It is a straight line against a curve, so it is only a close guide for small moves.

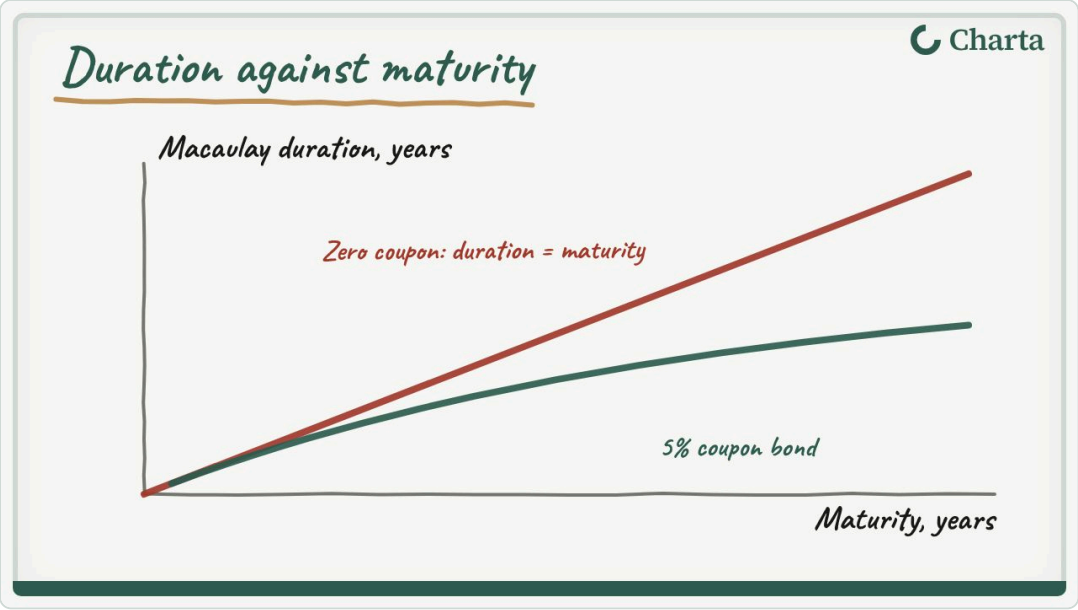
18 A story: the balance point

- Duration is the balance point of the payments
- Longer maturity: duration up
- Higher coupon: duration down
- Higher yield: duration down

IN THE LESSON

- Picture a long plank with a weight at each date a payment arrives.
- A big weight sits at the end for the final repayment, and little ones sit at each coupon date.
- The duration is the point where the plank would balance.
- Now play with it.
- Make the bond longer, and the big weight slides to the right, so the balance point moves right.
- Raise the coupon, and the little weights get heavier, pulling the balance point to the left.
- Raise the yield, and far away payments are shrunk the most by discounting, so again the balance point moves left.
- Those are the three rules: longer maturity raises duration, higher coupon lowers it, and higher yield lowers it.

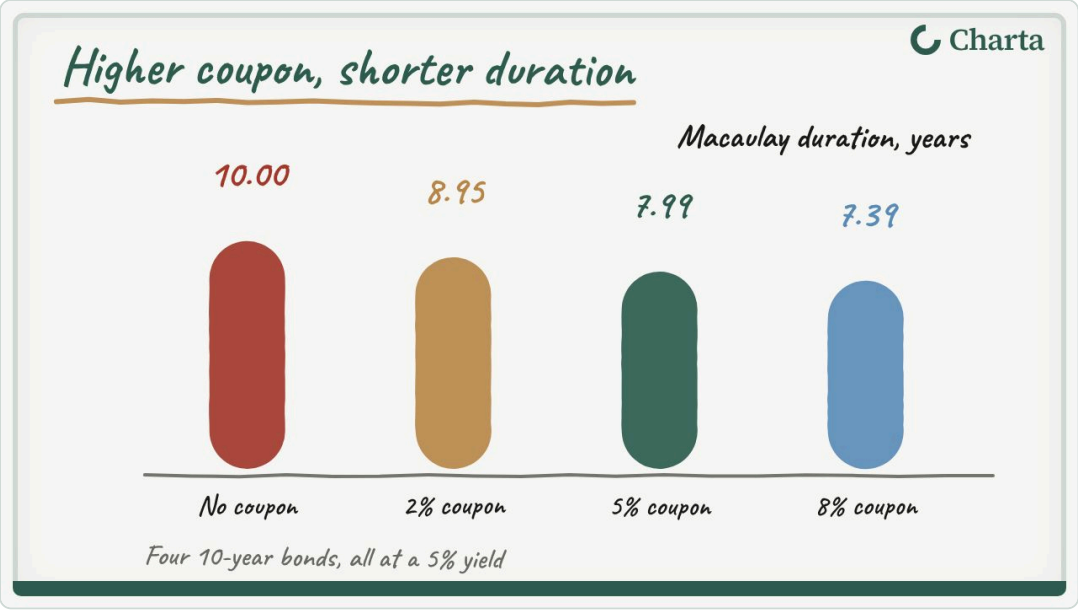
19 Duration against maturity



IN THE LESSON

- Here is the first rule as a picture.
- The bottom axis is the time to maturity, up to thirty years.
- The side axis is Macaulay duration in years.
- The straight line is a zero-coupon bond, whose duration equals its maturity at every point.
- Ten years of life gives a duration of ten years.
- The lower curve is a five per cent coupon bond at a yield of five per cent.
- At ten years its duration is seven point nine nine, below ten.
- As maturity grows the curve flattens, and at thirty years it is only fifteen point eight.
- So duration rises with maturity, but at a slowing pace, and a coupon bond always stays below the zero-coupon line.

20 Higher coupon, shorter duration



IN THE LESSON

- Now the second rule.
- Here are four ten-year bonds, all at a yield of five per cent, differing only in coupon.
- The first pays no coupon at all, so its Macaulay duration is exactly ten years.
- The second pays a two per cent coupon, and its duration drops to eight point nine five.
- The five per cent coupon bond, like Bond B, is at seven point nine nine.
- And the eight per cent coupon bond is shortest, at seven point three nine.
- Why?
- Every extra coupon brings value back sooner.
- The balance point shifts toward today.
- The same maturity, then, can hide very different risk, and the zero-coupon bond carries the most.

21 Example 6: the same bond at three yields

10-year bond, 5% coupon twice a year, three yields

Yield 3%	price 117.17	**8.17 years**
Yield 5%	price 100.0	**7.99 years**
Yield 7%	price 85.79	**7.8 years**

IN THE LESSON

- Example six covers the third rule.
- Take the ten-year five per cent coupon bond, and price it at three different yields.
- At three per cent it is a premium bond, priced at one hundred and seventeen point one seven, and its Macaulay duration is eight point one seven years.
- At five per cent, the price is one hundred and the duration is seven point nine nine years.
- At seven per cent, a discount bond at eighty-five point seven nine, the duration is seven point eight years.
- A higher yield shrinks the distant payments most, so the balance point moves left.
- The effect is gentle here, a fifth of a year, but it is always in this direction.

22 Example 7: which bond to own

Expect yields to fall 0.50%. Both 10-year bonds, yield 5%

Zero coupon, ModDur	10.00 ÷ 1.025	**9.76**
estimated gain	9.76 × 0.50%	**+4.88%**
8% coupon, ModDur	7.39 ÷ 1.025	**7.21**
estimated gain	7.21 × 0.50%	**+3.61%**

IN THE LESSON

- Example seven puts the rules to work.
- A portfolio manager expects yields to fall by fifty basis points, and may choose between two ten-year Dunmore bonds at a yield of five per cent: a zero-coupon bond and an eight per cent coupon bond.
- Which should she own?
- The zero's Macaulay duration is ten, and dividing by one point zero two five gives a modified duration of nine point seven six.
- The eight per cent bond has a Macaulay duration of seven point three nine and a modified duration of seven point two one.
- Fifty basis points down gives estimated gains of four point eight eight per cent and three point six one per cent.
- If she is right about falling yields, she picks the higher duration, the zero.

23 The rules of duration

What pushes duration up or down

If this rises	Duration
Maturity	goes up
Coupon rate	goes down
Yield	goes down
Zero-coupon bond	= maturity

IN THE LESSON

- Let us collect the rules.
- Longer maturity raises duration.
- A higher coupon lowers it.
- A higher yield lowers it.
- A zero-coupon bond sits at the limit: its Macaulay duration equals its maturity.
- And a bond with coupons has a Macaulay duration below its maturity, because some cash always arrives early.
- That gives you a quick check on any answer.
- If your Macaulay duration for a coupon bond is longer than its life, you have made an error.
- One footnote.
- For very long bonds sold at a deep discount, duration can peak before maturity, so the maturity rule is not perfectly strict.
- For ordinary bonds, these rules always hold.

24 A story: the price of one tick

- PVBP: money change for a 1 bp move in yield
- 1 bp = 0.01% = 0.0001
- In money, so it adds across positions
- The working measure for hedging

IN THE LESSON

- A trader on a bond desk does not think in percentages.
- She asks a simpler question: if yields move by the smallest step the market quotes, how many dollars do I make or lose?
- That smallest step is one basis point, one hundredth of one per cent.
- The answer, in money, is called the price value of a basis point, or P V B P.
- Some books call it the dollar value of an oh one.
- Because it is in money, it scales with the size of the position and adds up across a whole portfolio.
- That makes it the working measure for hedging, which means taking an opposite position so that losses on one side offset gains on the other.

25 PVBP by repricing

PVBP = (Price, yield down − Price, yield up) ÷ 2

Price, yield down 1bp	Reprice one basis point lower
Price, yield up 1bp	Reprice one basis point higher
PVBP	Per 100 of face; scale up

IN THE LESSON

- There are two ways to find it.
- The first is the direct one: reprice the bond twice.
- Lower the yield by one basis point and find the new price.
- Raise it by one basis point and find that price too.
- The price at the lower yield is higher, and the price at the higher yield is lower.
- Take the gap between them and halve it.
- Halving works out the average move for one basis point, up or down.
- This method uses no duration at all, only a pricing model, so it also works for strange bonds.
- The result is in currency per one hundred of face value, which you then scale to the position.

26 Example 8: PVBP of Bond B

Bond B, price 100.00 at 5.00%. Position: $10 million face value

Price at 4.99%	yield down 1 bp	**100.0780**
Price at 5.01%	yield up 1 bp	**99.9221**
PVBP per 100 face value	(100.0780 − 99.9221) ÷ 2	**0.07795**
Lots of 100 in the position	10,000,000 ÷ 100	**100,000**
PVBP of the position	0.07795 × 100,000	**$7,795**

IN THE LESSON

- Example eight.
- Reprice Bond B, the ten-year bond at five per cent.
- At a yield of four point nine nine per cent, which is one basis point lower, the price is one hundred point zero seven eight zero.
- At five point zero one per cent the price is ninety-nine point nine two two one.
- The gap is zero point one five five nine, and half of it is zero point zero seven seven nine five per one hundred of face value.
- Dunmore holds ten million of face value, which is one hundred thousand lots of one hundred.
- Multiply, and the PVBP of the position is about seven thousand seven hundred and ninety-five dollars.
- Each basis point of yield change moves the value by roughly that many dollars.

27 PVBP from money duration

PVBP ≈ ModDur × Full price × 0.0001

Modified duration	Percentage sensitivity
Full price of position	Position value, in currency
One basis point	0.01%, as a decimal

IN THE LESSON

- The second way uses what we already know.
- Money duration is the money change for a one percentage point move.
- A basis point is one hundredth of that, which is zero point zero zero zero one as a decimal.
- So the price value of a basis point is modified duration times the full price times zero point zero zero zero one.
- Another way to say it: PVBP is money duration scaled down by ten thousand.
- It is approximate, because duration is only a straight line, but for one basis point the difference is tiny.
- Notice the pattern: the repricing method is exact for a given pricing model, and the duration method is the quick check.

28 Example 9: the check, and a hedge

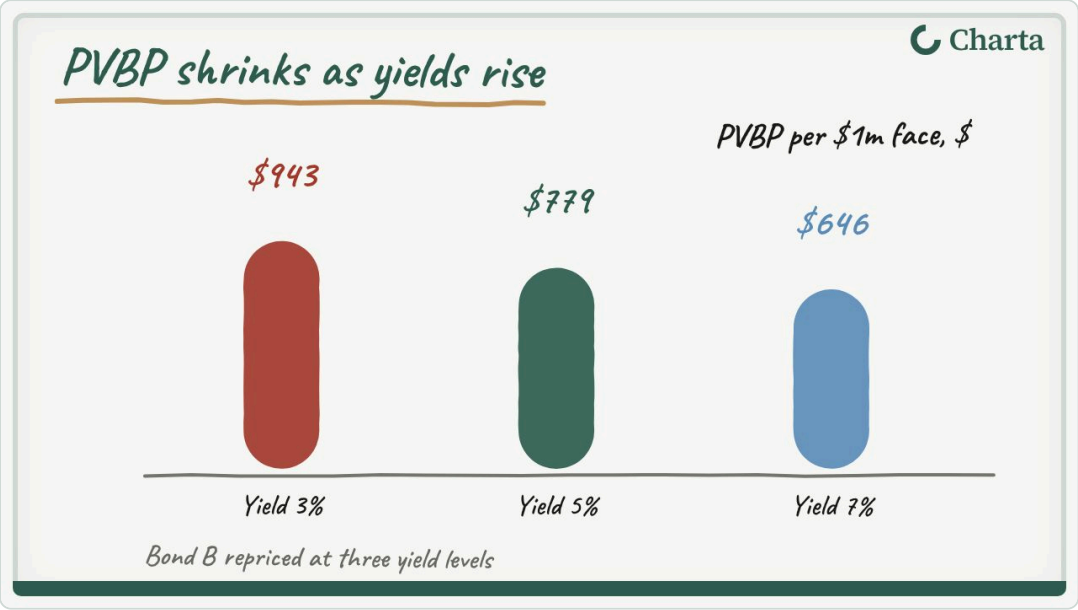
Bond B position, $10 million face value. Hedge bond: $450 per $1m face value

PVBP by duration	7.7946 × 10m × 0.0001	**$7,795**
Position PVBP	from repricing	**$7,795**
Hedge bond PVBP	per $1m face value	**$450**
Face value to short	7,795 ÷ 450	**$17.32m**

IN THE LESSON

- Example nine.
- Check the answer using duration.
- Bond B's modified duration, to four places, is seven point seven nine four six.
- Times ten million, times zero point zero zero zero one, gives seven thousand seven hundred and ninety-five dollars, the same as before, to the nearest dollar.
- Now the hedge.
- Suppose a hedging bond has a PVBP of four hundred and fifty dollars for each million of face value, an illustrative figure.
- To offset the position, short enough of it that its PVBP matches.
- Divide seven thousand seven hundred and ninety-five by four hundred and fifty.
- That is about seventeen point three two million of face value.
- A small parallel move in yields then roughly nets out.

29 PVBP shrinks as yields rise



IN THE LESSON

- Here is one more fact about PVBP.
- The bars show the PVBP of one million dollars of face value of Bond B, repriced at three different yield levels.
- At a yield of three per cent the bond trades well above par, and a basis point is worth about nine hundred and forty-three dollars.
- At five per cent it is about seven hundred and seventy-nine dollars.
- At seven per cent, with the bond trading below par, it is about six hundred and forty-six dollars.
- As yields rise, the price falls and the duration falls too, so PVBP shrinks twice over.
- That is why a hedge set up at one yield level needs adjusting after yields have moved.

30 Recap: the four measures

- %ΔPrice ≈ −ModDur × ΔYield, small moves
- ModDur = MacDur ÷ (1 + y/m)
- MoneyDur = ModDur × full price
- Duration up with maturity, down with coupon, yield
- PVBP = (P down − P up) ÷ 2 ≈ MoneyDur × 0.0001

IN THE LESSON

- Let us pull the lesson together.
- Modified duration estimates the percentage price change: minus modified duration times the change in yield, good for small moves, with the curve bending in the holder's favour for big ones.
- Macaulay duration is a weighted average time to the cash, in years, and modified duration is that divided by one plus the yield per period.
- Money duration is modified duration times the full price.
- Duration rises with maturity and falls with a higher coupon or a higher yield, and a zero-coupon bond sits at its maturity.
- And the price value of a basis point is the money change for one basis point, found by repricing or from money duration.

31 Try three quick questions

1. Mod. duration 6.00; yield up 25 bp. Estimated price change?
   Answer: −6.00 × 0.25% = -1.5%.
2. Semiannual bond, MacDur 7.00 years, yield 6%. Modified duration?
   Answer: 7.00 ÷ 1.03 = 6.80.
3. Position $4m, modified duration 5.0. PVBP?
   Answer: 5.0 × 4,000,000 × 0.0001 = $2,000.

IN THE LESSON

- Now try three quick questions yourself.
- Pause after each one.
- Question one.
- A bond has a modified duration of six.
- Its yield rises by twenty-five basis points.
- What is the estimated price change?
- Minus six times zero point two five per cent, which is minus one point five per cent.
- Question two.
- A bond that pays twice a year has a Macaulay duration of seven years and an annual yield of six per cent.
- What is its modified duration?
- Seven divided by one point zero three is six point eight.
- Question three.
- A position worth four million dollars has a modified duration of five.
- What is its price value of a basis point?
- Five times four million times zero point zero zero zero one is two thousand dollars.