Pricing options: Black-Scholes-Merton and Black

Derivatives

1 A story: a recipe needs a kitchen

- The formula is a recipe for a tidy market
- You must know the assumptions of that market
- Exam questions hide one broken rule
- European option: exercise only at expiry

IN THE LESSON

- Think of a recipe.
- It gives perfect results, but only in a kitchen that matches what the author had in mind: the right oven, the right ingredients.
- Change the kitchen and the recipe may fail.
- The Black Scholes Merton formula is like that.
- It gives one number for a European option, but only inside a tidy imaginary market.
- So the first skill is to know the kitchen.
- An exam question will often slip in one sentence that quietly breaks a rule, and you have to spot which rule it was.
- A European option can only be exercised on its expiry date, and that is the kind this formula prices.

2 The assumptions behind the formula

- Price follows geometric Brownian motion
- Constant, known volatility and risk-free rate
- Continuous, frictionless trading; shorting allowed
- No arbitrage; European exercise; known yield

IN THE LESSON

- Here is the list.
- One, the share price follows what is called geometric Brownian motion.
- In plain words, it moves randomly, and the size of each move is proportional to the price.
- Two, volatility, the spread of returns, is known and constant.
- Three, the risk free rate is known and constant.
- Four, trading is continuous and frictionless, so no taxes, no trading costs, and you can trade any amount at any moment.
- Five, short selling is allowed, and you may use all the proceeds.
- Six, there are no arbitrage opportunities.
- Seven, the option is European.
- And income from the underlying, such as dividends, enters as a known continuous yield.

3 What the model says about prices

Chart: Likelihood vs Price at expiry

- Floor at zero
- Long right tail

IN THE LESSON

- Look at the picture.
- The bottom line is the share price on the expiry date, from small to large.
- The vertical line shows how likely each price is.
- Follow the curve from the left.
- It starts at zero, because a share with limited liability can never be worth less than nothing.
- It rises quickly to a hump, then slides away to the right in a long tail.
- Big gains are possible, and the right tail stretches out a long way.
- Here is the point.
- The model does not make the price normal.
- It makes the continuously compounded return normal, and that makes the price lognormal: bounded below by zero, with a long right tail.

4 Example 1: why prices are lopsided

S = 100, volatility 20%, a 3 standard deviation move

Return, down 3 SD	−3 × 0.20	**−60%**
Price after the fall	100 × e^(−0.6)	**54.88**
Return, up 3 SD	+3 × 0.20	**+60%**
Price after the rise	100 × e^(0.6)	**182.21**

Symmetric returns, lopsided prices

IN THE LESSON

- Let us put numbers on it.
- Suppose a share is one hundred dollars, and the volatility is twenty per cent a year.
- To keep things simple we ignore the average drift.
- A move of three standard deviations downward means a continuously compounded return of minus sixty per cent, which is minus three times point two.
- Multiply the price by e to that power and you get fifty four point eight eight dollars.
- A move of the same size upward, plus sixty per cent, takes the price to one hundred eighty two point two one dollars.
- So the same sized move in the return gives a fall of about forty five dollars but a gain of about eighty two.
- The return is symmetric.
- The price is not, and it never goes negative.

5 Which rule is which? A quick sort

Learn this sorting: it is exam gold

Feature in the model	Behaviour
Continuously compounded returns	Normal
Share price at expiry	Lognormal
Volatility and risk-free rate	Constant
American call, no dividends	Same as European

IN THE LESSON

- Now sort the assumptions the way an exam would.
- Continuously compounded returns are normal.
- Prices are lognormal.
- Volatility is constant, and so is the risk free rate.
- Exercise is European, only at expiry.
- There is one more subtle point in your notes.
- The model also prices an American call on a share that pays no dividends, because early exercise is never optimal for it while interest rates are not negative.
- So the European label is not a limit for that special case.
- The trap, and it is a favourite, is to say that prices are normal, or that volatility may drift over time inside the model.
- Neither is true.

6 A story: buying a house with a mortgage

- A call acts like a house bought on a mortgage
- Term one: a holding of the underlying
- Term two: money borrowed to pay for it
- Together they replicate the call

IN THE LESSON

- Imagine you want to profit from a house that costs one hundred thousand, but you only put down a small deposit and borrow the rest.
- You hold the whole house, yet most of it is paid for by the loan.
- A call option behaves like that.
- The Black Scholes Merton call formula has two terms, and each one is a piece of that story.
- The first term is a holding of shares.
- The second is money borrowed to pay for most of those shares.
- Together they copy the call.
- That is why a dealer who sells a call can hedge it by building exactly that position, which is called a replicating portfolio.

7 The call formula, in two terms

c = S × e^(−δT) × N(d1) − X × e^(−rT) × N(d2)
d1 = [ln(S ÷ X) + (r − δ + σ²÷2) T] ÷ (σ√T); d2 = d1 − σ√T

Call value	What the call is worth
Share, less dividends	S × e^(−δT)
N of d1	Shares held per call
Strike, discounted	X × e^(−rT)
N of d2	Weight on the borrowing

IN THE LESSON

- Here is the formula.
- The call value is the share term minus the strike term.
- Start with the share term: the share price, multiplied by a dividend discount factor, e to the minus delta T, multiplied by N of d one.
- If there are no dividends, the factor is simply one.
- Now the strike term: the strike, discounted at the risk free rate, multiplied by N of d two.
- N means the standard normal cumulative probability, the chance that a standard normal draw lands below that number.
- And d one is the log of the share over the strike, plus the rate less the yield plus half the variance, all times the time, divided by volatility times the square root of time.
- Then d two is d one less volatility times the square root of time.

8 Example 2: price the call and read it

S = 100, X = 100, T = 1, r = 5%, σ = 20%, no dividend

d1 and d2	N = 0.6368 and 0.5596	**0.35, 0.15**
Shares held, value	0.6368 × 100	**$63.68**
Borrowing	95.12 × 0.5596	**$53.23**
Call price	63.68 − 53.23	**$10.45**

IN THE LESSON

- Time for our running case.
- A share is at one hundred, the strike is one hundred, there is one year to expiry, the risk free rate is five per cent, volatility is twenty per cent, and there are no dividends.
- First, d one is zero point three five, and d two is zero point one five.
- Next, N of d one is zero point six three six eight, and N of d two is zero point five five nine six.
- The share term is zero point six three six eight times one hundred, which is sixty three point six eight dollars.
- The strike discounted for a year is ninety five point one two dollars.
- Weighted by N of d two, the borrowing is fifty three point two three dollars.
- The call is therefore sixty three point six eight minus fifty three point two three, which is ten point four five dollars.

9 The call is mostly borrowed money

Dollars

Shares held	**63.68**
Borrowing	**53.23**
Call price	**10.45**

Shares held − borrowing = call price

IN THE LESSON

- Now see it as a picture.
- Three bars.
- The first bar is the share position, sixty three point six eight dollars of shares, which is the delta, N of d one, about zero point six three seven of one share.
- The second bar is the loan that pays for most of those shares, fifty three point two three dollars.
- The third bar is what is left: the call price of ten point four five dollars, which is what the buyer actually puts up.
- So the buyer controls sixty three point six eight dollars of shares with only ten point four five dollars of their own money.
- Divide one by the other and the leverage is about six point one times.
- If the share rises one per cent, the call rises roughly six per cent.
- And the mix is not fixed.
- It must be rebalanced as the price moves.

10 The mirror image, and the trap

- Put: short N(−d1) shares, lend PV(X) × N(−d2)
- Currency option: foreign rate takes the dividend's place
- Trap: borrowing is PV(X) × N(d2), not all of PV(X)
- N(d2) = risk-neutral chance of exercise

IN THE LESSON

- The put is the mirror image.
- A put is a short position in the underlying, N of minus d one shares, plus lending the present value of the strike, weighted by N of minus d two.
- So where a call buys shares with borrowed money, a put sells shares and lends the proceeds.
- For a currency option, the foreign risk free rate plays the role of the dividend yield.
- Now the trap.
- Students take the borrowing to be the whole present value of the strike, ninety five point one two dollars.
- Then the call would be sixty three point six eight minus ninety five point one two, which is negative, and a right can never be worth less than nothing.
- The strike is only paid if the call is exercised, so the loan is weighted by N of d two, the risk neutral chance of exercise.

11 A story: the rent you do not collect

- Owner collects the income, option holder does not
- Share option: slot holds the dividend yield δ
- Currency option: slot holds the foreign rate
- Domestic rate r always discounts the strike

IN THE LESSON

- Suppose you could buy an apartment, or just buy the right to buy it later.
- If you own the apartment you collect the rent every month.
- If you only hold the right, you collect nothing.
- So the right is worth a little less than the apartment suggests, and the missing rent is exactly the income you forgo.
- Shares work the same way.
- The holder of a share receives dividends, and the holder of an option does not.
- In the formula, we model that income as a continuous yield called delta.
- For a currency, the same slot is filled by the foreign interest rate, because anyone holding the foreign money earns that interest, and the option holder does not.

12 Example 3: a share paying a dividend yield

S = 50, X = 50, T = 1, r = 5%, σ = 20%

Call, no dividend		**$5.23**
Put, no dividend		**$2.79**
Call, 3% yield	50 × e^(−0.03) = 48.52	**$4.33**
Put, 3% yield	N(−d2) and N(−d1) form	**$3.37**

Dividend yield: call down, put up

IN THE LESSON

- Take a share at fifty dollars, a strike of fifty, one year, risk free rate five per cent, volatility twenty per cent.
- First with no dividend, the call is five point two three dollars and the put is two point seven nine.
- Now add a continuous dividend yield of three per cent.
- The share term shrinks: fifty times e to the minus point zero three is forty eight point five two.
- The strike term is fifty times e to the minus point zero five, which is forty seven point five six.
- With d one at zero point two zero zero and d two at zero point zero zero zero, the call falls to four point three three dollars and the put rises to three point three seven dollars.
- Dividends lower the call and raise the put, because the share drops by the amount paid out.

13 The yield pulls calls down and puts up

Option value ($)

Call, none	**5.23**
Call, 3%	**4.33**
Put, none	**2.79**
Put, 3%	**3.37**

One year, at the money, 20% volatility

IN THE LESSON

- Here are those four values side by side.
- The first two bars are the calls.
- With no dividend the call is five point two three dollars, and with the three per cent yield it falls to four point three three.
- The last two bars are the puts.
- With no dividend the put is two point seven nine, and with the yield it rises to three point three seven.
- The reason is the share term.
- The formula replaces today's price with a smaller one, the price less the value of the income paid out before expiry.
- A lower starting point hurts the right to buy and helps the right to sell.
- So if a question tells you a dividend is coming, expect the call to be lower and the put to be higher than without it.

14 Currencies: quote it the right way round

- Quote: domestic per one unit of foreign
- Domestic rate r discounts the strike
- Foreign rate sits where the dividend yield sat
- Foreign currency up, spot quote up, call gains

IN THE LESSON

- For currencies, the one rule is the quote.
- Express the spot rate as domestic currency per one unit of foreign currency.
- Then a call on the foreign currency is the right to buy a unit of foreign money at the strike, paid in the domestic currency.
- The domestic rate, r, discounts the strike.
- The foreign rate takes the place of delta in the share term.
- A call on the foreign currency gains when that currency strengthens, because the spot quote rises.
- Picture a United States investor with options on euros, quoted as dollars per euro.
- The dollar rate is domestic.
- The euro rate is foreign.
- If you ever feel unsure, ask which currency the strike is paid in.
- That currency's rate discounts the strike.

15 Example 4: a call on one million euros

S = 1.10 USD/EUR, X = 1.10, T = 0.5, σ = 10%, r(USD) = 4%, δ(EUR) = 2%

Euro leg	1.10 × e^(−0.02 × 0.5)	**1.08905**
Dollar leg	1.10 × e^(−0.04 × 0.5)	**1.07822**
d1 and d2	N = 0.57016, 0.54224	**0.177, 0.106**
Call per euro		**$0.0363**
On EUR 1m	0.0363 × 1,000,000	**$36,286**

IN THE LESSON

- A six month call on one million euros.
- Spot is one point one zero dollars per euro, the strike is the same, volatility is ten per cent, the dollar rate is four per cent and the euro rate is two per cent.
- The share term becomes the euro leg: one point one zero discounted at the euro rate, which is one point zero eight nine zero five.
- The strike term discounts at the dollar rate: one point zero seven eight two two.
- With d one at zero point one seven seven and d two at zero point one zero six, the call is one point zero eight nine zero five times zero point five seven zero one six minus one point zero seven eight two two times zero point five four two two four, which is zero point zero three six three dollars per euro.
- On one million euros that is about thirty six thousand three hundred dollars.

16 The trap: the two rates swapped

Same inputs, euro and dollar rates swapped

Version	Call on EUR 1m
Rates in place	$36,286
Rates swapped	$25,449
Error	−30%

IN THE LESSON

- Now watch the trap in action.
- Put the euro rate where the dollar rate belongs, and the dollar rate where the euro rate belongs, and the same inputs give twenty five thousand four hundred forty nine dollars instead of thirty six thousand two hundred eighty six.
- That is about thirty per cent too low, from one swap.
- Notice that the swapped call lands exactly on the correct put value, because spot equals the strike here.
- It is a good check.
- Remember the table.
- Dividend yield for shares; foreign rate for currencies; domestic rate always for the strike.
- And do not forget the dividend yield altogether.
- Leaving it out overstates the call and understates the put.

17 A story: the contract that costs nothing

- Futures cost nothing to enter
- Futures price already contains the carry
- Under risk-neutral pricing it does not drift
- So F replaces S, and F and X are discounted

IN THE LESSON

- Now a different underlying.
- Many options trade on futures, such as options on crude oil futures, on bond futures, or on index futures.
- A futures contract costs nothing to enter.
- You post margin, but you do not pay the price.
- So there is no money tied up and no income forgone, and the futures price already has the cost of carry built into it.
- Under risk neutral pricing, a futures price is not expected to grow at all.
- The Black model uses that fact.
- It takes the Black Scholes Merton idea, puts the futures price where the spot price was, and discounts both the futures price and the strike.

18 The Black model for options on futures

c = e^(−rT) × [F × N(d1) − X × N(d2)]
d1 = [ln(F ÷ X) + σ²T ÷ 2] ÷ (σ√T); c − p = e^(−rT)(F − X)

Discount factor	e^(−rT): discounts everything
Futures price F	Replaces the spot price
N of d1	N(d1)
Strike, discounted	Strike X, same bracket
N of d2	N(d2)

IN THE LESSON

- Here is the Black model.
- The call equals e to the minus r T, times the bracket F times N of d one, minus X times N of d two.
- The whole bracket is discounted once at the front, so both F and X are discounted.
- The put is e to the minus r T times the bracket X times N of minus d two, minus F times N of minus d one.
- And d one is the log of F over X, plus half the variance times T, all divided by volatility times the square root of T.
- Notice there is no r in d one.
- With a futures contract, there is no carry term.
- Then d two is d one minus volatility times the square root of T.
- Put call parity becomes call minus put equals e to the minus r T times F minus X.

19 Example 5: oil futures options

F = 82, X = 80, T = 0.5, σ = 30%, r = 4%

d1 and d2	σ√T = 0.2121	**0.2225, 0.010**
Discount factor	e^(−0.04 × 0.5)	**0.9802**
Call	e^(−rT) × (F N(d1) − X N(d2))	**$7.73**
Put	e^(−rT) × (X N(−d2) − F N(−d1))	**$5.77**
Parity	7.73 − 5.77 = e^(−0.02) × 2	**1.96**

IN THE LESSON

- A crude oil futures contract trades at eighty two dollars.
- Value six month European options struck at eighty, with volatility thirty per cent and a risk free rate of four per cent.
- Volatility times root T is zero point two one two one.
- So d one is zero point two two two five and d two is zero point zero one zero.
- The normal table gives N of d one as zero point five eight eight zero and N of d two as zero point five zero four one.
- The discount factor is e to the minus point zero two.
- The call is that factor times the difference between eighty two times zero point five eight eight zero and eighty times zero point five zero four one, which is seven point seven three dollars.
- The put, using the minus d values, is five point seven seven dollars.
- Check parity: call less put is one point nine six, and the discounted difference F minus X is the same.

20 Black against the misused spot formula

Option value ($)

Black call	**7.73**
Spot call	**8.72**
Black put	**5.77**
Spot put	**5.13**

Spot formula fed a futures price: call too high

IN THE LESSON

- Now the trap, in numbers.
- Suppose someone plugs the futures price of eighty two into the ordinary spot formula, discounting the strike but not the futures price.
- The call comes out at eight point seven two dollars, nearly a full dollar above the Black value of seven point seven three.
- The put comes out lower, at five point one three against five point seven seven.
- Why?
- The spot formula assumes the underlying is an asset you buy today, so it expects the price to grow at the risk free rate.
- A futures price is already a forward price and is not expected to grow.
- Discounting F as well as X removes that wrongly built in growth.
- So a question that hands you a futures price wants the Black model, not the spot formula.

21 Same bracket, different wrapping

Caplet

Rate set at reset → Black bracket on forward rate → × accrual × DF

Payer swaption

Option expires → Black bracket on swap rate → × annuity factor

IN THE LESSON

- The Black model reaches further than futures.
- Caplets, floorlets and swaptions are priced by treating a forward rate the way the Black model treats a futures price.
- Look at the top row, the caplet.
- A caplet is a call on a future interest rate.
- The rate is set on one date, the reset date, and the payment comes one period later.
- So the Black bracket is multiplied by the notional, by the fraction of a year that the rate accrues for, and by one discount factor.
- Now the bottom row, the swaption.
- Exercise delivers a whole swap that pays every period, so the bracket is multiplied by the notional and by the annuity factor.
- Only the wrapping changes.

22 Caplets, floorlets and swaptions

Caplet = N × accrual × DF × [FRA × N(d1) − X × N(d2)]
Payer swaption = N × annuity × [FS × N(d1) − X × N(d2)]

Notional	Notional amount
Accrual fraction	Fraction of a year
Discount factor	Discount factor, payment date
Annuity factor	Sum of discount factors
Black bracket	Rate times N(d1), less strike

IN THE LESSON

- Here are the two formulas.
- A caplet is worth the notional, times the accrual fraction, times the discount factor to the payment date, times the bracket: the forward rate agreement rate times N of d one, less the strike rate times N of d two.
- A floorlet uses the put form.
- A cap is simply the sum of caplets, one for each reset date, and a floor is a sum of floorlets.
- A payer swaption is the right to pay fixed.
- It is a call on the forward swap rate: the notional, times the annuity factor, times the bracket with the forward swap rate and the strike.
- A receiver swaption is the matching put.
- The annuity factor is the sum of the discount factors over the payment dates of the swap.

23 Example 6: one caplet

Notional $20m, 3-month rate, reset in 1y, FRA 3.2%, X 3.0%, σ 25%

d1 and d2	N = 0.6492, 0.5530	**0.3832, 0.1332**
Bracket	0.032 N(d1) − 0.030 N(d2)	**0.004185**
Wrapping	20m × 0.25 × 0.955	**4,775,000**
Caplet value	wrapping × bracket	**$19,985**

IN THE LESSON

- Take a caplet on a notional of twenty million dollars.
- The rate resets in one year and covers three months, so the accrual fraction is point two five.
- The forward rate is three point two per cent, the cap strike is three per cent, volatility is twenty five per cent, and the discount factor to the payment date is point nine five five.
- First d one is zero point three eight three two and d two is zero point one three three two.
- The bracket is point zero three two times zero point six four nine two minus point zero three times zero point five five three zero, which is zero point zero zero four one eight five.
- Multiply by the notional, the accrual fraction and the discount factor, and the caplet is worth about nineteen thousand nine hundred eighty five dollars.
- Leave out the accrual fraction and you would be four times too high.

24 Example 7: a payer swaption

$50m, pay 2.80%, forward swap rate 3.00%, σ 20%, DFs 0.940, 0.905, 0.870

Annuity factor	0.940 + 0.905 + 0.870	**2.715**
d1 and d2	N = 0.67183, 0.59676	**0.445, 0.245**
Bracket	0.030 N(d1) − 0.028 N(d2)	**0.0034456**
Swaption	50m × 2.715 × bracket	**$467,739**
No annuity	50m × bracket	**$172,280**

IN THE LESSON

- Now a swaption.
- A one year European payer swaption on fifty million dollars: the right to pay two point eight per cent fixed on a three year annual swap that starts in a year.
- The forward swap rate is three per cent and volatility is twenty per cent.
- The discount factors for years two, three and four are point nine four, point nine zero five and point eight seven.
- Add them: the annuity factor is two point seven one five.
- Then d one is zero point four four five, d two is zero point two four five, and the bracket is zero point zero zero three four four five six.
- The swaption is worth fifty million times two point seven one five times the bracket, about four hundred sixty seven thousand seven hundred dollars.
- Without the annuity factor you would price only one undiscounted year, about one hundred seventy two thousand three hundred dollars.

25 Four mistakes to avoid

- 1. Prices normal; volatility changing
- 2. Full PV of strike as the borrowing
- 3. Rates swapped; yield forgotten
- 4. F in spot formula; no accrual or annuity

IN THE LESSON

- Before the recap, the common slips.
- One.
- Saying prices are normal, or that volatility can change inside the model.
- Returns are normal, prices are lognormal, and volatility is constant.
- Two.
- Taking the borrowing in the replicating portfolio as the full present value of the strike.
- It is weighted by N of d two.
- Three.
- Swapping the domestic and foreign rates for a currency option, or ignoring the dividend yield for a share.
- Four.
- Plugging a futures price into the spot formula, or leaving out the accrual fraction of a caplet or the annuity factor of a swaption.
- Each of these produces a number that looks sensible but is wrong, and each is a standard exam trap.

26 Recap: the formulas to remember

BSM: c = S e^(−δT) N(d1) − X e^(−rT) N(d2)
Black: c = e^(−rT)[F N(d1) − X N(d2)]; swaption = N × annuity × bracket

Share, less dividends	Share less dividends
N of d2	Weight on borrowing
Futures price F	Futures price in Black
Accrual fraction	Accrual for a caplet
Annuity factor	Annuity for a swaption

IN THE LESSON

- Let us pull it together.
- The Black Scholes Merton call is the share term, with the dividend discount, times N of d one, minus the discounted strike times N of d two.
- That is a leveraged holding of the share, borrowing weighted by N of d two.
- A dividend yield lowers the call and raises the put; for a currency, the foreign rate takes the yield's place and the domestic rate discounts the strike.
- The Black model puts the futures price in place of the spot price and discounts the whole bracket.
- For rates, a caplet scales the bracket by the notional, the accrual fraction and one discount factor, and a payer swaption scales it by the notional and the annuity factor.

27 Try three quick questions

1. Futures F = 50, X = 48, r = 3%, T = 0.5. What is c − p?
   Answer: e^(−0.015) × 2 = $1.97
2. Under Black-Scholes-Merton, what is normally distributed?
   Answer: Continuously compounded returns; prices are lognormal.
3. Caplet: 10m × 0.5 × 0.97 × bracket 0.004. Value?
   Answer: $19,400

IN THE LESSON

- Now try three questions yourself.
- Pause after each one.
- Question one.
- A call and a put on a futures contract share the same strike and expiry.
- The futures price is fifty, the strike is forty eight, the rate is three per cent and the time is half a year.
- What is the call minus the put?
- It is the discounted difference, e to the minus point zero one five, times two, about one point nine seven dollars.
- Question two.
- Under the Black Scholes Merton model, which quantity is normally distributed?
- The continuously compounded return, so the price is lognormal.
- Question three.
- A caplet has a notional of ten million, an accrual fraction of a half, a discount factor of point nine seven and a bracket of point zero zero four.
- What is its value?
- Multiply them to get nineteen thousand four hundred dollars.
- Well done.