Binomial interest rate trees and bond values

Fixed Income, Level Two

1 A story: pricing the bag item by item

- A straight bond: coupons plus principal, no options
- Each payment has its own date and its own spot rate
- Discount each payment at its own spot rate, then add
- The result is the arbitrage-free value

IN THE LESSON

- Rohan Mehta is asked a simple question: what is a plain three year bond worth?
- Think of a bag of groceries.
- You could ask the shop for one price for the whole bag, or you could price each item on its own and add them up.
- Bond valuation is the second way.
- A straight bond, one with no options attached, is just a bundle of dated payments: a coupon each year, and at the end the principal.
- Money due in one year and money due in three years are priced differently in the market, and the price for each date is its spot rate.
- So Rohan discounts each payment at the spot rate for its own date.
- The result is called the arbitrage free value, because no trader could make a riskless profit against it.

2 The value of a straight bond

$$V_0 = CF_1/(1+S_1) + CF_2/(1+S_2)^2 + \ldots + (CF_n+Par)/(1+S_n)^n$$

Cash flow	Coupon at each date
Spot rate	Rate for that date only
Years to payment	The exponent
Principal	Added to the last coupon
Value today	Sum of discounted pieces

IN THE LESSON

- Here is the rule.
- The value today is the first cash flow divided by one plus the one year spot rate, plus the second cash flow divided by one plus the two year spot rate, squared, and so on out to the last date.
- Take each piece.
- The cash flow is the coupon at every date, and at the last date it also includes the principal.
- Every date has its own spot rate.
- The exponent is the number of years to that payment.
- Add up the discounted pieces and you have the value today.

3 Example 1: a three-year 4% bond

Spot rates 2.0%, 2.8%, 3.3%. Coupon 4%, face value 100

Year 1	$4 \div 1.02$	**3.9216**
Year 2	$4 \div 1.028^2$	**3.7851**
Year 3, with principal	$104 \div 1.033^3$	**94.3479**
Value of the bond	sum of the three	**102.0546**

IN THE LESSON

- Example one.
- The spot rates are two per cent for one year, two point eight per cent for two years and three point three per cent for three years.
- The bond pays a four per cent annual coupon on a face value of one hundred.
- Year one: four divided by one point oh two is three point nine two one six.
- Year two: four divided by one point oh two eight, squared, is three point seven eight five one.
- Year three includes the principal, so it is one hundred and four divided by one point oh three three, cubed, which is ninety four point three four seven nine.
- Add them up and the value is one hundred and two point oh five four six.
- That is above par, because the coupon of four is higher than every spot rate.

4 Most of the value sits in year three

Present value

Year 1	**3.92**
Year 2	**3.79**
Year 3	**94.35**

The three add up to 102.05

IN THE LESSON

- Here are those three present values as bars.
- Year one contributes three point nine two, year two contributes three point seven nine, and year three contributes ninety four point three five.
- Almost all of the value sits in the last payment, because that payment holds the principal.
- That is why the three year spot rate does most of the work in this valuation.
- But the two small bars still matter.
- They are discounted at their own, lower rates, and getting them wrong shifts the total.
- Every spot rate here is below the four per cent coupon, which is why the bond is worth more than its face value.

5 Where the spot rates come from

- Strip prices give spot rates directly
- Bootstrapping: solve the par curve date by date
- A spot rate is a chain of one-year forward rates
- Forward route and spot route give the same value

IN THE LESSON

- Where do the spot rates come from?
- There are two routes.
- The first is strip prices.
- A strip is a single payment from a government bond, sold on its own as a zero coupon bond, and its price gives the spot rate for that date directly.
- The second route is bootstrapping from the par curve.
- You take the one year par rate, then use the two year par bond to solve for the two year spot rate, and so on.
- Either way you reach the same spot curve.
- There is also a third view.
- A spot rate is a chain of one year forward rates.
- So discounting through the forward rates, one year at a time, gives exactly the same value.
- The next example shows it.

6 Example 2: the same value through forwards

Same bond, now through one-year forward rates

Forward for year 2	$1.028^2 \div 1.02 - 1$	**3.61%**
Forward for year 3	$1.033^3 \div 1.028^2 - 1$	**4.31%**
Year 3 payment today	$104 \div (1.02 \times 1.036063 \times 1.043073)$	**94.3479**
Total value	same as the spot route	**102.0546**

IN THE LESSON

- Example two checks the value another way.
- The one year forward rate for year two is one point oh two eight squared, divided by one point oh two, minus one.
- That is three point six one per cent.
- The one year forward rate for year three is one point oh three three cubed, divided by one point oh two eight squared, minus one, which is four point three one per cent.
- Now chain them.
- The year three payment of one hundred and four is discounted back through the forwards: divide by one plus the year three forward, then by one plus the year two forward, then by one point oh two.
- That gives ninety four point three four seven nine, exactly as before.
- The other payments follow, and the total is again one hundred and two point oh five four six.

7 Example 3: why it is called arbitrage-free

Market quote 101.50 against a value of 102.0546

Buy the bond at the quote	pay	**101.50**
Strip it, sell three pieces	3.9216+3.7851+94.3479	**102.0546**
Riskless profit	sales minus cost	**0.5546**
If quoted above the value	sell bond, buy strips	**reverse**

IN THE LESSON

- Example three shows where the name comes from.
- Suppose the same bond is quoted in the market at one hundred and one point five oh, below Rohan's value.
- He can buy the bond for one hundred and one fifty, strip it into its three payments, and sell each one as a zero coupon bond at the price the spot rates imply.
- Those three sales bring in one hundred and two point oh five four six.
- The profit is about fifty five cents, with no risk.
- Traders doing this push the bond price up until the gap closes.
- If the bond were quoted above its value, the trade runs the other way: sell the bond, buy the strips.
- So the arbitrage free value is the only price at which neither trade pays.

8 Example 4: the single-rate slip

Same bond, every payment discounted at 3.3%

Own spot rate for each date	correct	**102.0546**
Every payment at 3.3%	the slip	**101.9687**
Error	too low by	**0.0859**
Coupon against spot rates	4% beats 2.0, 2.8, 3.3	**premium**

IN THE LESSON

- Now the classic slip.
- Suppose a junior colleague discounts every payment at three point three per cent, the three year spot rate.
- Four divided by one point oh three three, plus four divided by one point oh three three squared, plus one hundred and four divided by one point oh three three cubed, comes to one hundred and one point nine six eight seven.
- That is zero point oh eight five nine too low.
- It looks small, but across a large portfolio it adds up.
- The error exists because the two early coupons are discounted at a rate meant for three year money, which is higher than the one and two year rates.
- The same slip happens if you borrow another bond's yield.
- Always match each date with its own spot rate.

9 A story: a fork in the path

- A tree maps the possible one-year rates at each date
- Up and down moves are equally likely
- Volatility, σ, sets how far apart neighbours sit
- Lognormal model: rates stay positive

IN THE LESSON

- So far the rates were certain.
- Now suppose Rohan must value a bond that can be called.
- Its cash flows depend on where interest rates go, and a single spot curve cannot say that.
- He needs a map of possible future rates.
- At each step, rates can go up or go down, each equally likely.
- Draw all the forks and you have a binomial interest rate tree.
- A node is a point on the tree where the rate takes one particular value.
- A date is a point in time, and at each date the tree shows the possible one year rates.
- How far apart the rates sit is set by volatility, the annual standard deviation of rate changes, written sigma.
- Level Two uses a lognormal model.

10 The ratio between neighbouring rates

Upper rate = Lower rate × e^(2σ)

Upper rate	Rate at the higher node
Lower rate	Rate at the lower node
σ: volatility	Annual rate volatility
Ratio between neighbours	e^(2σ), same all through

IN THE LESSON

- Here is the rule that builds the tree.
- At any one date, the rate at an upper node equals the rate at the node just below it, times e raised to two sigma.
- Turn it round and the lower rate equals the upper rate times e raised to minus two sigma.
- Take the pieces.
- The letter e is about two point seven one eight.
- Sigma is the annual volatility, for example fifteen per cent.
- So two sigma is thirty per cent, and the ratio between neighbours is e raised to zero point three, about one point three five.
- Notice it is a ratio, not a number of points added.
- That is what keeps rates positive, however many down moves occur.

11 Example 5: filling in the tree

Volatility 15%. Lowest rates: 3.00% at date 1, 2.80% at date 2

Ratio between neighbours	e^(2 × 0.15)	**1.3499**
Date 1, upper node	3.00% × 1.3499	**4.05%**
Date 2, middle node	2.80% × 1.3499	**3.78%**
Date 2, top node	3.78% × 1.3499	**5.10%**

IN THE LESSON

- Example five.
- The tree has fifteen per cent volatility.
- The rate today is two per cent.
- A year from now the lower rate is three per cent, and two years from now the lowest rate is two point eight per cent.
- Rohan fills in the rest.
- The ratio is e to the power of two times zero point one five, which is one point three four nine nine.
- The upper rate at date one is three per cent times one point three four nine nine, which is four point oh five.
- The middle rate at date two is two point eight times one point three four nine nine, which is three point seven eight.
- The top rate at date two is three point seven eight times the ratio, five point one oh.
- Three inputs fix the tree: the rate today, the lowest rate at each later date, and the volatility.

12 The tree drawn out

Chart: One-year rate (%) vs Date: today, year 1, year 2

- 2.00%
- 4.05%
- 3.00%
- 5.10%
- 3.78%
- 2.80%

IN THE LESSON

- Here is that tree drawn out.
- The date runs along the bottom, from today on the left to two years from now on the right.
- The one year rate goes up the side.
- Start at the left, with today's rate of two per cent.
- From there the path forks.
- A year from now the rate is either three per cent or four point oh five per cent.
- Follow each of those.
- From the lower node, rates go to two point eight or three point seven eight.
- From the upper node, they go to three point seven eight or five point one oh.
- Two routes lead to the middle node on the right, and they land on the same rate.
- So date two has three nodes, not four.

13 The gaps widen as rates rise

Gap to neighbour (points)

Date 1	**1.05**
Date 2, lower	**0.98**
Date 2, upper	**1.32**

Same ratio, bigger gap at higher rates

IN THE LESSON

- Look at the gaps between neighbouring rates, as bars.
- At date one, the gap between three per cent and four point oh five is one point oh five points.
- At date two, the gap between the lower pair, two point eight and three point seven eight, is zero point nine eight.
- And between the upper pair, three point seven eight and five point one oh, it is one point three two.
- The ratio is the same each time, one point three five.
- But the gap in points grows as the level rises.
- That is the meaning of lognormal in practice.
- When rates are high they swing by more points, and when they are low by fewer.
- And rates can never fall below zero, because multiplying by a positive ratio never gives a negative number.

14 Counting nodes and paths

- At date n there are n + 1 nodes
- Recombining: up then down equals down then up
- Paths double each date: 2, 4, 8
- Middle node at date 2: probability ½

IN THE LESSON

- Now some bookkeeping that appears in exam questions.
- At date zero there is one node.
- At date one there are two.
- At date two there are three.
- In general, at date n there are n plus one nodes.
- The tree is recombining: an up move followed by a down move reaches the same node as a down followed by an up.
- That keeps the tree small.
- The number of paths still doubles, so there are two paths to date one, four to date two, and eight to date three.
- Because each move is equally likely, the middle node at date two is reached by two of the four paths, so it carries a probability of one half.
- That is why the middle rates reflect the centre of the distribution.

15 Example 6: using e^σ instead of e^(2σ)

Lower date-1 rate 3.00%, volatility 15%

Correct ratio	e^0.30	**1.3499**
Correct upper rate	3.00% × 1.3499	**4.05%**
Slip ratio	e^0.15	**1.1618**
Slip upper rate	3.00% × 1.1618	**3.49%**

IN THE LESSON

- Example six is the classic slip.
- Rohan's colleague builds the date one pair using e to the power of sigma instead of two sigma.
- With fifteen per cent volatility, e to the power of zero point one five is one point one six one eight.
- The lower rate is three per cent, so the upper rate comes out at three point four nine per cent.
- The right answer is four point oh five.
- The gap is zero point four nine points instead of one point oh five.
- The tree now carries only about half the volatility that Rohan intended.
- For a callable bond, that would understate the value of the call option, and so overstate the value of the bond.
- Check that the ratio is e to the power of two sigma.

16 A story: tuning the piano

- Benchmarks: usually par bonds priced at 100
- Calibrate: choose node rates that reprice every benchmark
- Then the tree shows no arbitrage in those bonds
- The rates are not forecasts

IN THE LESSON

- Rohan now has a way to draw a tree, but which rates go in it?
- Any tree he draws must agree with the market, or it would give silly prices.
- Think of tuning a piano.
- You do not play whatever notes you like.
- You adjust each string until it matches a reference tone.
- In a tree, the reference tones are benchmark bonds, usually par bonds, whose market price is one hundred.
- Calibrating the tree means choosing the node rates so that the tree prices each benchmark at exactly its market price.
- Then the tree shows no arbitrage in those bonds.
- The rates are not forecasts.
- They are simply the set of rates that makes the tree agree with today's market.

17 How calibration proceeds

Calibrating to par bonds

One-year rate known → Fix the volatility → Search for the lowest rate → Hold, then next date

IN THE LESSON

- Calibration goes one date at a time.
- Follow the boxes.
- First, the one year rate is known from the market, in our example two per cent.
- Second, fix the volatility.
- That sets the ratio between neighbouring nodes at every date.
- Third, for the next date, search for the lowest rate that makes the next benchmark bond worth exactly one hundred when valued backward through the tree.
- The upper rate follows from the ratio.
- Fourth, hold those rates fixed and move to the next date, using the next par bond.
- Repeat maturity by maturity until the tree is full.
- The search is trial and error, and software does it in a flash.
- What matters is knowing which way to move when the answer misses.

18 Example 7: searching for the rate

Par rates 2% and 3%. Volatility 15%. Two-year 3% bond

Trial: 3.00% and 4.05%	above 100: rates too low	**100.49**
Trial: 4.00% and 5.40%	below 100: rates too high	**99.39**
Solved: 3.44% and 4.65%	the bond prices at par	**100.00**

IN THE LESSON

- Example seven.
- The one year par rate is two per cent.
- The two year par rate is three per cent, so a two year bond with a three per cent coupon is worth one hundred.
- The volatility is fifteen per cent, so the upper rate at date one is one point three four nine nine times the lower.
- Rohan guesses a lower rate of three per cent, giving an upper rate of four point oh five.
- Working back, the bond comes to one hundred point four nine, above one hundred.
- The tree discounts too lightly, so the rates are too low.
- Next guess: four per cent and five point four oh.
- Now the bond is ninety nine point three nine, so these rates are too high.
- The answer lies between, at three point four four and four point six five.

19 The search drawn as a curve

Chart: Bond value vs Trial lower rate at date 1 (%)

- 3.00%: 100.49
- 3.44%: 100.00
- 4.00%: 99.39

IN THE LESSON

- The same search is drawn here.
- Along the bottom is the trial lower rate at date one, from two and a half to four and a half per cent.
- Up the side is the value of the two year bond, worked back through the tree.
- The dashed line is par, one hundred.
- Now the falling curve.
- Higher rates mean heavier discounting, so a higher guess always gives a lower value.
- That is why there is one answer, and you always know which way to go.
- At three per cent the curve is above par, at one hundred point four nine.
- At four per cent it is below par, at ninety nine point three nine.
- It crosses par at three point four four per cent.
- That is the calibrated lower rate, and the upper rate is four point six five.

20 Example 8: checking the benchmark

Check: the two-year 3% bond on the calibrated date-1 rates

Lower node, 3.442%	103 ÷ 1.03442	**99.57**
Upper node, 4.646%	103 ÷ 1.04646	**98.43**
Average, coupons added	½×102.57 + ½×101.43	**102.00**
Discount at 2.00%	102.00 ÷ 1.02	**100.00**

IN THE LESSON

- Rohan checks the answer by revaluing the benchmark.
- At the lower node, the last payment is one hundred and three, discounted at three point four four two per cent, the rate to three decimals: one hundred and three divided by one point oh three four four two is ninety nine point five seven.
- At the upper node, the rate is four point six four six per cent, and one hundred and three divided by one point oh four six four six is ninety eight point four three.
- Step back to today.
- Add the coupon of three to each value, average them to get one hundred and two, then divide by one point oh two.
- The answer is exactly one hundred.
- The tree reprices its benchmark.

21 Higher volatility: wider, still fitted

Date-1 rate (%)

15% low	**3.44**
15% high	**4.65**
25% low	**3.06**
25% high	**5.04**

Both trees reprice the benchmark at 100

IN THE LESSON

- Now change the volatility.
- At fifteen per cent, the date one rates were three point four four and four point six five.
- Raise the volatility to twenty five per cent, and the ratio becomes e to the power of zero point five, about one point six five.
- Rohan repeats the search, and the pair becomes three point oh six and five point oh four.
- Look at the bars: the pair sits further apart.
- Both trees price the benchmark at exactly one hundred, because both were calibrated to it.
- So volatility does not change the fit to the benchmarks.
- It changes the value of anything with an option inside, which is the whole reason we build a tree rather than rely on spot rates alone.

22 Mid-lesson check: four slips to avoid

- One rate for every payment, not the matching spot rate
- e^σ instead of e^(2σ) for the ratio
- Above 100 means rates too low: move them up
- Node rates as forecasts; volatility moving a par price

IN THE LESSON

- Before we use the tree, here are the slips so far.
- First, discounting every payment at one rate, such as the longest spot rate or another bond's yield, instead of the matching spot rate for each date.
- Second, using e to the power of sigma instead of e to the power of two sigma, which halves the volatility in the tree.
- Third, moving the rates the wrong way after a miss: a value above the benchmark price means the rates are too low and must go up.
- Fourth, reading calibrated node rates as forecasts, or expecting volatility to move the price of a calibrated par bond.
- It will not, because the tree is fitted to it.

23 A story: working back from the cabin

- Start at maturity, where the cash flow is certain
- Step back one date at a time
- At each node: average, add cash flows, discount
- The value at today's node is the bond's value

IN THE LESSON

- Now we use the tree to value a bond.
- Imagine a trail that ends at a cabin, and at each fork you can only ask what lies one step ahead.
- You start at the cabin, where everything is known, and work back to the trailhead.
- That is backward induction.
- At maturity the cash flow is known for certain: the final coupon and the principal.
- At the date before, each node looks ahead to two possible next nodes, takes the average of their values plus whatever cash flows are paid, and discounts for one year at its own rate.
- Repeat until you reach today.
- The value at today's node is the value of the bond.
- Rohan will value his three year four per cent bond on the tree from the framework section.
- That tree is only illustrative and is not calibrated to the spot curve, so its value need not match the spot value.

24 The node value

V = [½(V↑ + C↑) + ½(V↓ + C↓)] ÷ (1 + r)

V: value at this node	The value we are finding
V up, V down	Values at the next two nodes
C: cash flow paid	Coupon due at the next date
One half each	Up and down equally likely
r: this node's rate	Discount rate: this node's

IN THE LESSON

- Here is the rule.
- The value at a node is a half times the value of the up node plus its cash flow, plus a half times the value of the down node plus its cash flow, all divided by one plus this node's own rate.
- The two halves are there because up and down are equally likely.
- The cash flow is the coupon due at the date of the next nodes, and at the last date it includes the principal.
- We add the cash flow to the value before averaging, because the coupon is paid on that date and belongs to whoever holds the bond.
- Finally, we divide by one plus the rate at this node, not another.

25 Backward induction, with and without options

Straight bond

Start at maturity → Add coupons → Average, discount → Reach today

Bond with an option

Value the node → Compare with price → Call: keep lower → Then step back

IN THE LESSON

- The method is drawn here for two kinds of bond.
- In the top row, a straight bond.
- Start at maturity, where the cash flow is certain.
- Move back one date.
- At every node, add the coupon to each of the two next values, average them, and divide by one plus that node's rate.
- Keep going until today.
- In the bottom row, a bond with an option.
- Add one check at each exercise date.
- Compare the node value with the call price or the put price.
- A call lets the issuer pay one hundred and be done, so you keep the lower of the two.
- A put lets the holder sell at one hundred, so you keep the higher.
- Then step back.
- We will see a call at the end.

26 Example 9: date two, then date one

Three-year 4% bond. Date-2 nodes carry the final 104

Date 2, top node, 5.10%	104 ÷ 1.0510	**98.95**
Date 2, middle, 3.78%	104 ÷ 1.0378	**100.21**
Date 2, bottom, 2.80%	104 ÷ 1.0280	**101.17**

IN THE LESSON

- Example nine.
- Rohan values the three year bond, four per cent annual coupon, face value one hundred, on the tree from before.
- Rates are two per cent today, then four point oh five or three per cent, then five point one oh, three point seven eight or two point eight.
- Start at date two, one year before maturity.
- Only the last payment of one hundred and four is left.
- Top node: one hundred and four divided by one point oh five one is ninety eight point nine five.
- Middle node: one hundred and four divided by one point oh three seven eight is one hundred point two one.
- Bottom node: one hundred and four divided by one point oh two eight is one hundred and one point one seven.
- Date one comes next.

27 Example 9: date one and today

Add the coupon of 4 to each next value, average, discount

Date 1, upper, 4.05%	(½×102.95+½×104.21) ÷ 1.0405	**99.55**
Date 1, lower, 3.00%	(½×104.21+½×105.17) ÷ 1.03	**101.64**
Today, 2.00%	(½×103.55+½×105.64) ÷ 1.02	**102.54**

IN THE LESSON

- At date one, the upper node at four point oh five per cent leads to ninety eight point nine five and one hundred point two one.
- Add the coupon of four to each, average, and divide by one point oh four oh five.
- That gives ninety nine point five five.
- The lower node, at three per cent, leads to one hundred point two one and one hundred and one point one seven.
- Do the same, divide by one point oh three, and the value is one hundred and one point six four.
- For today, add the coupon again, average, and divide by one point oh two.
- The answer is one hundred and two point five four.

28 The values on the tree

Chart: Node rate (%): bond value vs Date: today, year 1, year 2

- 2.00%: 102.54
- 4.05%: 99.55
- 3.00%: 101.64
- 5.10%: 98.95
- 3.78%: 100.21
- 2.80%: 101.17

IN THE LESSON

- Here are all those values on the tree, with the rate at each node and the bond's value beside it.
- Read it from the right.
- At date two, on the right, the values are ninety eight point nine five at the top, one hundred point two one in the middle and one hundred and one point one seven at the bottom.
- Higher rate, lower value.
- Step left to date one.
- The upper node is worth ninety nine point five five and the lower node one hundred and one point six four.
- Step left again to today, and the value is one hundred and two point five four.
- The values at each date are before that date's coupon, which is why the coupon is added as we move left.

29 Example 10: the two standard mistakes

Same bond and tree. Value today if done correctly: 102.54

Correct	ends with ÷ 1.02	**102.54**
Coupons left out	no 4s added	**94.83**
Last discount forgotten	no ÷ 1.02 at the end	**104.60**
Shortfall without coupons	102.54 − 94.83	**7.71**

IN THE LESSON

- Example ten shows the two standard mistakes, and what each does to the answer.
- The correct value is one hundred and two point five four.
- First mistake: leaving out the coupons at the intermediate nodes.
- The answer falls to ninety four point eight three, about seven point seven too low.
- Second mistake: forgetting the last discounting step, from date one back to today.
- Then he has a value one year from now, not today.
- That gives one hundred and four point six oh, too high by roughly the one year rate.
- Too low by several points suggests missing coupons.
- Too high by about two per cent suggests a missing final discount.

30 Example 11: the same bond, callable at 100

Call at 100 on either date. Cap each node at 100

Date 2 after the cap	98.95, then 100 and 100	**2 capped**
Date 1, upper node	(½×102.95+½×104) ÷ 1.0405	**99.45**
Date 1, lower: 100.97 capped	104 ÷ 1.03, then cap	**100.00**
Today	(½×103.45+½×104) ÷ 1.02	**101.69**
Value of the call	102.54 − 101.69	**0.85**

IN THE LESSON

- Finally, a bond with an option.
- Suppose the issuer may call the same bond at one hundred at date one or date two, after the coupon is paid.
- At every node, Rohan compares the value with one hundred, and keeps the lower.
- At date two, the middle and bottom values are above one hundred, so both are replaced by one hundred.
- At date one, the upper node is now ninety nine point four five, below one hundred, so it stays.
- The lower node works out at one hundred point nine seven, so it is replaced by one hundred.
- Today, the value is one hundred and one point six nine.
- The option free value was one hundred and two point five four, so the call is worth about eighty five cents to the issuer.

31 Recap: from spot rates to a tree value

Spot values → tree → calibrate → work back

Spot values	Each cash flow, own spot rate
Tree rates	Ratio e^(2σ), n + 1 nodes
Calibration	Par bonds must price at 100
Backward induction	Average V + C, ÷ (1 + r)

IN THE LESSON

- Let us put it together.
- Spot values: discount each cash flow at its own spot rate, and add up.
- The tree: neighbouring rates at a date differ by e to the power of two sigma, with n plus one nodes at date n.
- Calibration: choose the node rates so that every benchmark par bond comes out at one hundred, a value above one hundred meaning rates too low.
- Backward induction: start at maturity, add the coupon to each next value, average, divide by one plus the node's rate, and for a call or put compare with the exercise price at each date.
- Do not skip the coupons, or the last discount.

32 Try three quick questions

1. Spots 2%, 3%. Two-year 5% bond. Value?
   Answer: 5 ÷ 1.02 + 105 ÷ 1.03² = 103.87
2. Volatility 10%, lower rate 2.5%. Upper rate?
   Answer: 2.5% × e^0.2 = 3.05%
3. Next values 98 and 96, coupon 5, rate 4%. Node value?
   Answer: (½×103 + ½×101) ÷ 1.04 = 98.08

IN THE LESSON

- Now try three quick questions yourself.
- Question one.
- Spot rates are two per cent for one year and three per cent for two years.
- What is a two year bond with a five per cent annual coupon worth?
- Five divided by one point oh two plus one hundred and five divided by one point oh three squared is one hundred and three point eight seven.
- Question two.
- Volatility is ten per cent and the lower rate is two and a half per cent.
- What is the upper rate?
- Two and a half per cent times e to the power of zero point two is three point oh five per cent.
- Question three.
- The next two values are ninety eight and ninety six, the coupon is five, and the node rate is four per cent.
- What is the node value?
- Half of one hundred and three plus half of one hundred and one is one hundred and two, divided by one point oh four, which is ninety eight point oh eight.