The yield curve and spot rates

Fixed Income

1 A case: Meera and the committee

- Meera Okafor, analyst at Brindle Capital
- Each wait has its own price: a rate per maturity
- The list of rates is the yield curve
- From it: costs, expectations, manager choices

IN THE LESSON

- Meet Meera Okafor, an analyst at a made up firm called Brindle Capital.
- Her committee asks a simple question: what is the government bond market telling us?
- Meera knows that one number cannot answer it.
- A bond pays money in a year, in two years, in three, and the market charges a different rate for each wait.
- That list of rates, from short to long, is the yield curve.
- From the curve she can read what each wait costs, what the market expects for next year, and what a manager could earn by choosing one bond over another.
- We follow her case all lesson.

2 Spot rates, and the shape of the curve

- Spot rate: today's yield on one payment on one date
- Plot spot rates by maturity: the spot curve
- Shapes: upward, flat, inverted (downward)
- Our case: 1.5, 2.5, 3.2, 3.7, 4.0% for 1 to 5 years

IN THE LESSON

- A spot rate is the yield today on a single payment due on one date, such as a zero coupon bond.
- It is the cleanest rate there is, because nothing else is mixed in.
- Plot the spot rates against maturity and you have the spot curve.
- Its shape matters.
- It can slope upward, which is normal, with longer waits paying more.
- It can be flat.
- Or it can slope downward, which we call inverted.
- Our case uses a rising curve.
- The one year spot rate is one point five per cent, two years is two point five, three years is three point two, four years is three point seven and five years is four.

3 Three curves from one set of prices

Chart: Rate (%) vs Maturity (years)

- Forward
- Spot
- Par yield

IN THE LESSON

- Here are three curves drawn from the same bond prices.
- Maturity in years runs along the bottom and the rate goes up the side.
- The middle curve is the spot curve, rising from one point five per cent to four.
- The gold curve underneath is the par curve, the yield on bonds that pay a coupon and sell at face value.
- Notice it sits just below the spot curve.
- The red curve above is the one year forward rate for each year, the rate for borrowing during that year only.
- It climbs fastest and stays on top.
- On a rising curve, the order from top to bottom is forwards, then spots, then par yields.

4 Forward rates from spot rates

$(1 + S2)^2 = (1 + S1) \times (1 + 1y1y)$
$1y1y = (1 + S2)^2 \div (1 + S1) - 1$

Later spot rate	Two-year spot: S2
Earlier spot rate	One-year spot: S1
Forward rate	1y1y: year-two rate, set today

IN THE LESSON

- A forward rate is an agreed rate for a future period, and today's spot curve already fixes it.
- Think of two routes to the same date.
- Route one, lock in the two year spot rate.
- Route two, lock in the one year spot rate now and then roll into whatever the one year rate is a year later.
- If the second route could beat the first for sure, everyone would do it, so prices adjust until they match.
- That means one plus the two year spot, squared, equals one plus the one year spot, times one plus the one year, one year forward rate.
- Rates are compounded, never averaged.

5 Example 1: forward rates from the curve

Spots: 1y 1.5%, 2y 2.5%, 3y 3.2%

1y1y forward	$1.025^2 \div 1.015 - 1$	**3.51%**
2y1y forward	$1.032^3 \div 1.025^2 - 1$	**4.61%**
1y2y forward	$(1.032^3 \div 1.015)^{\wedge}½ - 1$	**4.06%**

Trap: do not average spot rates; compound them

IN THE LESSON

- Example one.
- Using Meera's curve, find three forward rates.
- The one year, one year forward is one point zero two five squared, divided by one point zero one five, minus one.
- That is three point five one per cent.
- The two year, one year forward starts in two years and lasts one year: one point zero three two cubed, divided by one point zero two five squared, minus one, which is four point six one per cent.
- Last, the one year, two year forward is a two year rate starting in a year.
- Divide one point zero three two cubed by one point zero one five, take the square root, subtract one, and you get four point zero six per cent.
- See that this lies between the first two forwards.

6 Yield to maturity is a blend

- Yield to maturity: one rate for the whole bond
- Price = all cash flows at that one rate
- Rising curve: bond yield sits below its spot rate
- Same maturity, different coupon, different yield

IN THE LESSON

- Now the third number.
- Spot rates price one payment at a time.
- But a bond pays a stream: coupons, then the face value at the end.
- Its yield to maturity is the single discount rate that makes the price equal to all of those payments together.
- So it is a blend of the spot rates, weighted toward the dates where most of the money arrives.
- This has two consequences.
- First, on a rising curve, a coupon bond's yield sits below the spot rate for the same maturity, because its early coupons are discounted at the low short rates.
- Second, two bonds with the same maturity but different coupons have different yields.

7 Example 2: price a bond, then find its yield

3-year bond, 4% annual coupon, face value 100

Year-one coupon	$4 \div 1.015$	**3.94**
Year-two coupon	$4 \div 1.025^2$	**3.81**
Final payment	$104 \div 1.032^3$	**94.62**
Price	sum of the three	**102.37**
Yield to maturity	solve for one rate	**3.16%**

IN THE LESSON

- Example two.
- Brindle holds a three year bond with a four per cent annual coupon.
- First price it with the spot curve, discounting each payment at its own spot rate.
- The first coupon is four divided by one point zero one five, which is three point nine four.
- The second is four divided by one point zero two five squared, about three point eight one.
- The final payment, one hundred and four, divided by one point zero three two cubed, is ninety four point six two.
- The price is one hundred and two point three seven.
- Now find the single rate that gives the same price.
- By trial and error it is three point one six per cent, a little under the three point two per cent spot rate.

8 Is the yield the return you will earn?

- Yield = expected return only if all three hold:
- 1. Held to maturity 2. No default
- 3. Every coupon reinvested at that same yield
- Reinvest lower: return below the yield

IN THE LESSON

- Here is the exam favourite.
- Does a bond earn its yield to maturity?
- Only under three conditions.
- One, you hold it to maturity.
- Two, it does not default.
- Three, and this is the one people forget, every coupon is reinvested at that same yield.
- The yield quietly assumes the cash from each coupon can be put to work at the bond's own yield rate.
- Nothing in the market promises that.
- If reinvestment rates turn out lower, the realised return falls below the yield.
- If higher, it rises above.
- Expected return and promised yield are close cousins, never twins, and the gap is reinvestment.

9 Example 3: realised return on a two year bond

2-year bond, 4% coupon, held to maturity

Price today	4 ÷ 1.015 + 104 ÷ 1.025²	**102.93**
Yield to maturity	solve for one rate	**2.48%**
Reinvest coupon at 3.51%	4 × 1.0351 + 104	**108.14**
Return, reinvest at 3.51%	(108.14 ÷ 102.93)^½ − 1	**2.50%**
Return, reinvest at 1.00%	(108.04 ÷ 102.93)^½ − 1	**2.45%**

IN THE LESSON

- Example three.
- Take a two year bond with a four per cent coupon.
- Priced with the spot curve, it costs one hundred and two point nine three, and its yield is two point four eight per cent.
- Now suppose the market does what the forward rate says, so the coupon is reinvested at three point five one per cent.
- At the end you hold one hundred and eight point one four.
- Turn that into an annual return: it is exactly two point five per cent, the two year spot rate.
- Not the yield.
- If instead the coupon is reinvested at only one per cent, you hold one hundred and eight point zero four, and the return is two point four five per cent, below the yield.

10 Next: spot rates from par yields

- Market quotes par yields; pricing needs spot rates
- Bootstrapping climbs one maturity at a time
- Year one: spot rate equals the par yield
- Each next spot rate uses all the earlier ones

IN THE LESSON

- A problem.
- In practice, markets quote par yields, because government bonds are issued at face value with a coupon.
- But to price a single payment, or a strange cash flow, we need spot rates.
- How do we get from one to the other?
- We bootstrap.
- Picture climbing a ladder.
- You cannot reach the third rung without standing on the second, and you cannot stand on the second without the first.
- The one year spot rate comes straight from the one year par yield.
- Then each next rate is solved using all the rates below it.
- The word bootstrap means exactly that, pulling yourself up by what you already have.

11 The bootstrapping equation

100 = Σ coupon ÷ (1 + Sk)^k + (100 + coupon) ÷ (1 + Sn)^n
(1 + Sn)^n = (100 + c) ÷ (100 − earlier PVs)

Coupon	Par coupon = par yield
Earlier coupons, discounted	Each at its own spot rate
Spot rate being found	Solved last, root of n

IN THE LESSON

- Here is the logic written down.
- A par bond is priced at one hundred.
- So one hundred equals each coupon discounted at its own spot rate, plus the coupon and face value at the end, discounted at the last spot rate.
- All spot rates are known except the last.
- So take the earlier coupons, discount each at its own spot rate, and subtract them from one hundred.
- What is left is the value today of the final payment.
- Divide the final payment by that remainder, take the root that matches the number of years, and subtract one.
- Out comes the spot rate for that maturity.
- Then repeat for the next.

12 Example 4: bootstrap the two year rate

Par yields: 1y 1.8%, 2y 2.6%, 3y 3.2%

S_1	single cash flow	**1.800%**
Year-one coupon	2.6 ÷ 1.018	**2.554**
Left for the last payment	100 − 2.554	**97.446**
$(1 + S_2)^2$	102.6 ÷ 97.446	**1.0529**
S_2	square root, minus one	**2.610%**

IN THE LESSON

- Example four.
- Meera sees these annual par yields: one year, one point eight per cent; two years, two point six; three years, three point two.
- Start at the bottom.
- The one year spot rate is one point eight per cent, because a one year bond has a single cash flow.
- Now the two year par bond: coupon two point six, price one hundred.
- The first coupon, discounted at one point eight per cent, is two point five five four.
- Take that from one hundred and you have ninety seven point four four six, the value today of the final payment of one hundred and two point six.
- Divide, take the square root, subtract one: two point six one per cent.

13 Example 4 continued: the three year rate

Three-year par bond, coupon 3.2%

Coupon one, discounted	3.2 ÷ 1.018	**3.14**
Coupon two, discounted	3.2 ÷ 1.0261^2	**3.04**
Left for the last payment	100 − 6.18	**93.82**
$(1 + S_3)^3$	103.2 ÷ 93.82	**1.1000**
S_3	cube root, minus one	**3.228%**

IN THE LESSON

- Now the third rung.
- The three year par bond has a coupon of three point two.
- It pays that coupon at years one and two, then one hundred and three point two at year three.
- Strip out the early coupons using the spot rates we just found.
- The year one coupon is three point two over one point zero one eight, about three point one four.
- The year two coupon is three point two over one point zero two six one squared, about three point zero four.
- Together, six point one eight.
- Subtract from one hundred to get ninety three point eight two.
- Divide one hundred and three point two by that, take the cube root, subtract one, and the three year spot rate is three point two two eight per cent.

14 Spot rates against par yields

Par yield to spot rate, in per cent

Maturity	Par → spot
One year	1.80 → 1.80
Two years	2.60 → 2.61
Three years	3.20 → 3.23

IN THE LESSON

- Put the results side by side.
- At one year, the spot rate and the par yield are the same, one point eight per cent.
- At two years, the par yield is two point six and the spot rate two point six one.
- At three years, the par yield is three point two and the spot rate three point two two eight.
- The gap grows with maturity.
- Why does the spot rate come out higher?
- On a rising curve, the early coupons of the par bond are discounted at low rates, so they are worth more, leaving less for the final payment.
- For the bond to still be worth one hundred, the final payment must be discounted at a higher rate.

15 Mistakes that cost marks so far

- Averaging spot rates, instead of compounding
- Yield taken as a guaranteed return
- Forgetting to strip out the earlier coupons
- Wrong exponent on the earlier coupons

IN THE LESSON

- Let us pause and collect the traps.
- First, averaging spot rates instead of compounding them: the forward rate comes from dividing growth factors, not from subtracting rates.
- Second, treating the yield to maturity as the return you will certainly earn.
- It is only that if coupons are reinvested at the yield.
- Third, in bootstrapping, forgetting to strip out the earlier coupons, and treating all of one hundred as the price of the final payment.
- Fourth, using the wrong exponent: the coupon at year two is divided by the year two spot rate squared, not by the rate once.
- Check each one against your last answer.

16 What do active managers believe?

- Horizon: one year; choose bill or longer bond
- Forwards are the path already in prices
- On that exact path, every bond earns the same
- Active view = future spot rates differ from forwards

IN THE LESSON

- Now put yourself in a manager's chair.
- You have one year to invest.
- You can buy a one year bill, or a two year bond and sell it in a year.
- Which earns more?
- You may think the answer depends on whether rates rise.
- It does not, at least not in the way you expect.
- The market has already built a path for rates into the forward curve.
- Following that path exactly, every bond earns the same over your horizon, whatever its maturity.
- So the forward rate is the break-even, and an active manager is simply someone who thinks future spot rates will differ from the forwards.

17 The one year return on a longer bond

Return = $(1 + S2)^2 \div (1 + \text{new } S1) - 1$
If new S1 = 1y1y, return = today's S1

Two-year spot today	S2: known today
One-year spot in a year	Unknown: your forecast
Break-even: the forward	1y1y: the market's view

IN THE LESSON

- Here is the algebra for a one year horizon.
- You buy a two year zero coupon bond today and sell it in a year.
- At sale it is a one year bond, valued at the one year spot rate that exists then, which we call the new spot rate.
- Your return is one plus the two year spot, squared, divided by one plus the new spot rate, minus one.
- Now set the new spot rate equal to the one year, one year forward.
- The two cancel to leave exactly the one year spot rate, today's bill rate.
- That is the break-even.
- A new spot rate below it beats the bill.
- Above it loses.

18 One year returns on the two year zero

One-year return (%)

New rate 2.5%	**2.50%**
New rate 3.51%	**1.50%**
New rate 4.5%	**0.54%**
One-year bill	**1.50%**

The bill earns 1.50% in every case

IN THE LESSON

- Here are the results as bars.
- We buy the two year zero coupon bond at two point five per cent and sell it a year later, when it is a one year bond.
- The first bar: the new one year rate turns out to be two point five per cent.
- Rates rose, from one point five.
- Yet the zero returns two point five per cent.
- The second bar: the new rate equals the forward, three point five one, and the return is one point five, exactly the bill.
- The third bar: the new rate jumps to four point five, and the return drops to just over a half of one per cent.
- The bill stays at one point five per cent in all three cases.

19 Example 5: extend or stay short?

Spots 1.5% and 2.5%; forward 3.51%; horizon one year

Two-year zero today	$100 \div 1.025^2$	**95.18**
Price in a year at 3.0%	100 ÷ 1.03	**97.09**
Return, view of 3.0%	97.09 ÷ 95.18 − 1	**2.00%**
Return, view of 4.5%	95.69 ÷ 95.18 − 1	**0.54%**
Decision at 3.0%	view below 3.51% forward	**extend**

IN THE LESSON

- Example five.
- Meera has a one year horizon.
- The two year zero coupon bond costs one hundred divided by one point zero two five squared, which is ninety five point one eight.
- The forward rate is three point five one per cent.
- Her view: in a year the one year rate will be three per cent.
- Then the bond sells for one hundred over one point zero three, ninety seven point zero nine, a return of two per cent, against one point five on the bill.
- Her view is below the forward, so she extends.
- Had she expected four and a half per cent instead, the bond would return only point five four per cent, below the bill, so she would stay short.

20 The trap: compare with forwards, not spot

- Do not compare your view with today's spot rate
- Compare it with the forward rate
- View below forward: extend maturity
- View above forward: stay short

IN THE LESSON

- The most common slip is to compare the view with today's spot rate.
- The manager expects the one year rate to rise from one point five to three.
- Rising rates, so stay short, right?
- Wrong.
- The market has already priced in a rise to three point five one.
- Three is a smaller rise than the market allowed for, so the longer bond comes out ahead.
- The reverse also holds: a manager expecting rates to fall can still lose by extending, if they fall by less than the forwards imply.
- The rule is clean.
- View below the forward, extend.
- View above the forward, stay short.
- Spot rates today do not enter.

21 Rolling down the curve

- Belief: the curve stays where it is
- Buy a bond longer than the horizon
- It ages: maturity shortens, yield falls
- Lower yield means higher price: roll down gain

IN THE LESSON

- Now a strategy for a manager who believes something quite different: that the curve will simply stay where it is.
- Picture a hill with a bond walking down it.
- Each year the bond gets one year older, so its maturity shortens.
- On a rising curve, shorter maturities have lower yields, so as the bond ages it slides down the curve to a lower yield, and a lower yield means a higher price.
- That extra price gain, on top of the interest, is the roll down return.
- The recipe: buy a bond with a maturity longer than your horizon, hold it to the horizon, and sell it as a shorter bond.

22 Sliding down the spot curve

Chart: Spot rate (%) vs Maturity (years)

- Sell: 2.5%
- Buy: 3.2%

IN THE LESSON

- Here is the idea drawn.
- Maturity runs along the bottom, the spot rate up the side, and the green curve is today's rising spot curve.
- The red label marks where we buy: the three year bond, at three point two per cent.
- Our horizon is one year.
- A year later, with the curve unchanged, the same bond is a two year bond, so it sits on the curve at two years, at two point five per cent.
- The gold label shows where we sell.
- The bond has moved left and down the curve, from three point two to two point five.
- That drop in yield, seven tenths of a point, is what lifts the price.

23 The roll down return

Return = (P1 + coupons) ÷ P0 − 1
P1 uses today's yield for the shorter maturity

Price paid today	Longer bond, its own yield
Price at the horizon	Same bond, aged, shorter yield
Rolled return	Compare with yield and bill

IN THE LESSON

- The calculation is a plain price return.
- You buy at the price today, using the yield for the bond's full maturity.
- You sell at the horizon, using the yield for the shorter maturity the bond has aged into, as read off today's curve.
- The return is the selling price divided by the buying price, minus one.
- If the bond pays a coupon, add the coupon received over the year to the selling price.
- The key assumption sits in the second price: the curve at the horizon is the same as the curve today.
- Change that, and you have a different strategy, one that bets on rates moving.

24 Example 6: roll down a three year zero

Curve unchanged: 1y 1.5%, 2y 2.5%, 3y 3.2%

Buy 3-year zero	$100 \div 1.032^3$	**90.98**
Sell as 2-year zero	$100 \div 1.025^2$	**95.18**
One-year return	$95.18 \div 90.98 - 1$	**4.61%**
Yield at purchase	3.2%	**3.20%**
One-year bill	matures at the horizon	**1.50%**

IN THE LESSON

- Example six.
- Meera has a one year horizon and believes the curve will stay put.
- She buys the three year zero coupon bond at three point two per cent.
- It costs one hundred over one point zero three two cubed, which is ninety point nine eight.
- A year later, with an unchanged curve, it is a two year zero priced at two point five per cent: one hundred over one point zero two five squared, ninety five point one eight.
- Her return is ninety five point one eight over ninety point nine eight, minus one, which is four point six one per cent.
- That beats the three point two per cent yield she started with, and beats the one point five per cent bill by more than three points.

25 Example 7: when the shape is different

Same bought bond: 3-year zero at 90.98

Flat curve, 2y at 3.2%	$93.89 \div 90.98 - 1$	**3.20%**
Downward, 2y at 3.8%	$92.81 \div 90.98 - 1$	**2.01%**
Upward, 2y at 2.5%	from Example 6	**4.61%**
Break-even 2y yield	price must reach 92.35	**4.06%**

Flat: nothing to roll. Downward: rolls up

IN THE LESSON

- Example seven tests the assumption.
- If the curve is flat at three point two, the bond ages into a two year yield of three point two, sells at ninety three point eight nine, and returns three point two per cent.
- Exactly its yield.
- Nothing to roll.
- If the curve slopes down at these maturities, and the two year yield is three point eight, the bond sells at ninety two point eight one and returns just over two per cent, below its yield.
- It has rolled up.
- Last, how much cushion does the upward curve give?
- The bond still matches the bill, as long as the two year yield at the horizon stays below four point zero six per cent.

26 Three ways to spend one year

One-year return (%)

One-year bill	**1.50%**
Two-year, rolled	**3.51%**
Three-year, rolled	**4.61%**

Curve unchanged over the year

IN THE LESSON

- Here is the comparison.
- All three bars are one year returns on the same curve, if it stays put.
- The first is the bill, which matures at the horizon and earns one point five per cent.
- The second is the two year zero, bought at two point five per cent.
- A year later it is a one year bond at one point five, so it is sold at a higher price, and the return is three point five one.
- The third is the three year zero, rolled down to two years, four point six one per cent.
- The longer the bond you buy, the more curve it slides down, so long as the curve keeps rising at those points.

27 Roll down is a bet on stability

- Roll down expects spot rates below the forwards
- Curve stable and upward sloping: gain
- Flat curve: nothing to roll down
- Curve shifts up far enough: gain wiped out

IN THE LESSON

- Let us connect this to the last idea.
- The forward curve says rates will climb: it implies a one year rate of three point five one a year from now.
- Rolling down assumes the opposite, that the curve stays put and the one year rate is still one point five.
- In forward language, the roller expects future spot rates below the forwards, which is exactly the view that says to extend.
- The risk is plain too.
- If the curve shifts up by enough, the price falls and wipes out the gain.
- So it is not a free lunch.
- It rewards a stable, upward sloping curve, and punishes a flat or downward one.

28 Recap of the whole lesson

- Spot prices one payment; forward comes by compounding
- Yield equals return only with reinvestment at yield
- Bootstrap: strip earlier coupons, then solve
- View vs forward decides; roll down needs a still curve

IN THE LESSON

- Let us pull it together.
- Spot rates price single payments.
- Forward rates come from compounding the spot curve, never from averaging it.
- The yield to maturity is one blended rate, and it equals the return you earn only if you hold to maturity, nothing defaults and coupons are reinvested at that yield.
- Bootstrapping turns par yields into spot rates one maturity at a time, stripping out the earlier coupons first.
- Active managers measure their view against the forwards: below, extend; above, stay short.
- Rolling down earns extra when the curve is upward sloping and stays still.

29 Try three quick questions

1. Spots: 1y 3%, 2y 3.5%. What is the 1y1y forward?
   Answer: $1.035^2 \div 1.03 - 1 = 4.00\%$
2. Par yields: 1y 2%, 2y 3%. Find the 2y spot rate.
   Answer: (103 ÷ 97.06)^½ − 1 ≈ 3.02%
3. A manager expects spot rates above the forwards. Action?
   Answer: Stay short: longer bonds fall below break-even

IN THE LESSON

- Question one.
- The one year spot rate is three per cent and the two year spot rate is three point five per cent.
- What is the one year, one year forward rate?
- Answer: one point zero three five squared, divided by one point zero three, minus one, which is four per cent.
- Question two.
- Par yields are two per cent at one year and three per cent at two years.
- What is the two year spot rate?
- Answer: the first coupon is worth three over one point zero two, so what is left of one hundred is ninety seven point zero six.
- A hundred and three over that, square root, minus one, is about three point zero two per cent.
- Question three.
- A manager expects future spot rates above the forwards.
- What does she do?
- Answer: stay short, because a longer bond would fall short of the break-even.