CHARTA LEARNING · LESSON NOTES

Risk aversion, the efficient frontier, the capital allocation line and the market portfolio

CFA Level I · Portfolio Management

You will learn: How to score portfolios with a utility formula, find the minimum variance and efficient frontiers, draw the capital allocation line, and explain why every investor holds the tangency portfolio, which is the market portfolio.

Running example: Examples follow a made-up club with two funds: the Harbor Stock Fund (expected return 12%, standard deviation 20%) and the Bridge Bond Fund (5%, 8%), correlation 0.2. The risk-free rate is 3%.

1. Risk aversion and utility

- A risk averse investor prefers less uncertainty at the same expected return and needs extra expected return to accept extra risk. It does not mean refusing all risk. Risk neutral ignores risk; risk seeking likes the gamble. Finance usually assumes risk aversion.
- Utility squeezes return and risk into one score. A is the risk aversion coefficient: the bigger A, the bigger the penalty. The penalty uses variance, not standard deviation, so doubling the risk quadruples it. A risk-free asset's utility is just its return.
- An indifference curve joins all risk and return combinations that give one investor the same utility. It slopes upward and bends up faster as risk grows. Higher and to the left is better, one investor's curves never cross, and a more risk averse investor has steeper curves.
- The best portfolio is where the highest reachable indifference curve just touches the frontier.

$$U = E(R) - \tfrac{1}{2} \times A \times \sigma^2$$

THREE PORTFOLIOS (RETURN / RISK): MOSTLY BONDS 5.56% / 7.84%, HALF AND HALF 8.50% / 11.49%, ALL STOCK 12% / 20%

A = 4: mostly bonds	0.0556 − 2 × 0.006144	**0.0433**
A = 4: half and half	0.085 − 2 × 0.0132	**0.0586 (chosen)**
A = 4: all stock	0.12 − 2 × 0.04	**0.0400**
A = 10: mostly bonds / half / all stock	0.0556 − 5 × 0.006144 etc.	**0.0249 (chosen) / 0.0190 / −0.0800**
A = 1: half / all stock	0.085 − 0.5 × 0.0132; 0.12 − 0.5 × 0.04	**0.0784 / 0.1000 (chosen)**

2. Minimum variance and efficient frontiers

- The minimum variance frontier shows the lowest risk reachable for each expected return from blends of the assets. Its leftmost point is the global minimum variance portfolio. With many assets the possible portfolios fill a region and the frontier is its left edge.
- A portfolio dominates another if it gives more return for the same risk, or less risk for the same return. The lower branch is always dominated, so nobody holds it.
- The efficient frontier is only the part from the global minimum variance portfolio upward. Along it, no point beats another without a trade, so the investor's own taste picks the point.
- The minimum variance formula needs variances, not standard deviations: square the risks first. Adding a little of a riskier asset can make the whole portfolio safer because the two partly cancel.

> **w(stock) = (σ²bond − Cov) ÷ (σ²stock + σ²bond − 2 × Cov), with Cov = ρ × σstock × σbond**

LEAST RISKY BLEND, AND A DOMINATED PORTFOLIO

Covariance	0.2 × 0.20 × 0.08	**0.0032**
Top / bottom	0.0064 − 0.0032; 0.04 + 0.0064 − 0.0064	**0.0032 / 0.0400**
Stock weight	0.0032 ÷ 0.04	**8% (return 5.56%, risk 7.84%)**
25% stock / 50% stock		**risk 8.54%, return 6.75% / risk 11.49%, return 8.50%**
All bonds vs 16% stock, 84% bonds	both risk 8.00%; returns 5.00% and 6.12%	**all bonds dominated by 1.12 points**

Watch out: Calling the whole minimum variance frontier efficient, when only the upper part is.

3. How correlation bends the frontier

- Return does not depend on correlation; risk does. At correlation +1 the frontier is a straight line (risk is the weighted average, no gain from mixing). At 0.2 it bows to the left. At −1 it is a sharp V whose tip reaches zero risk.
- The lower the correlation, the further the frontier bows. An asset that moves differently can shift the whole frontier to the left even if its own return is ordinary.

50 : 50 MIX, RISKS 20% AND 8%, RETURN 8.50% IN EVERY CASE

Correlation +1	0.5 × 20% + 0.5 × 8%	**14.00%**
Correlation 0.2	square root of 0.0132	**11.49%**
Correlation −1	0.5 × 20% − 0.5 × 8%	**6.00%**

4. The capital allocation line

- The risk-free asset has a sure return, zero risk and zero covariance with anything. Mixing it with a risky portfolio makes the teamwork term vanish, so the combinations lie on a straight line, the capital allocation line.
- If w is below one you lend at the risk-free rate (hold cash); if above one you borrow and invest more than your own money. Return and risk both scale with w.
- The line's intercept is the risk-free rate and its slope is the Sharpe ratio: extra return per unit of risk. The steepest line just touches the frontier at the tangency portfolio, the risky portfolio with the highest Sharpe ratio.
- A cautious investor should not hunt for a safer risky portfolio: keep the tangency portfolio and hold more cash.

E(Rp) = Rf + w × (E(Rt) − Rf); σp = w × σt; slope = (E(Rt) − Rf) ÷ σt

RISK-FREE 3%; RISKY PORTFOLIO IS HALF AND HALF (8.5%, 11.49%)

40% risky, 60% cash	return 0.4 × 8.5 + 0.6 × 3; risk 0.4 × 11.49%	**5.20%; 4.60%**
150% risky, borrow 50%	return 1.5 × 8.5 − 0.5 × 3; risk 1.5 × 11.49%	**11.25%; 17.23%**
Sharpe: minimum variance mix	(5.56 − 3) ÷ 7.84	**0.327**
Sharpe: all stock	(12 − 3) ÷ 20	**0.450**
Sharpe: half and half	(8.5 − 3) ÷ 11.49	**0.479 (highest)**
At risk 4.6%: via half / via min-var	3 + 0.4 × 5.5; 3 + 0.587 × 2.56	**5.20% / 4.50%**

Watch out: Confusing the tangency portfolio with the minimum variance portfolio; the minimum variance point is the leftmost point.

5. Two-fund separation and the market portfolio

- The decision splits in two. Which risky portfolio? The tangency portfolio, the same for everyone with the same expectations and one risk-free rate. How much of it? That depends on risk aversion. The two funds are the risk-free asset and the tangency portfolio.
- If everyone holds the same risky portfolio and together they own every risky asset, that portfolio holds every risky asset in proportion to its market value. In equilibrium the tangency portfolio is the market portfolio.
- Replacing the tangency portfolio with the market portfolio turns the capital allocation line into the capital market line, with total risk (standard deviation) on the horizontal axis. Its slope is the Sharpe ratio of the market. The security market line, met later, puts beta there.
- Assumptions: shared expectations, borrowing and lending at one risk-free rate, and tradeable, divisible assets with no taxes or costs. If they fail, separation fails, so treat the theory as a benchmark.

Capital market line: E(Rp) = Rf + [(E(Rm) − Rf) ÷ σm] × σp

THREE INVESTORS, ONE TANGENCY MIX (HALF STOCK FUND, HALF BOND FUND)

Cautious (40% risky)		**20% stock, 20% bonds, 60% cash**
Moderate (100% risky)		**50% stock, 50% bonds**
Bold (150% risky)		**75% stock, 75% bonds, borrow 50%**
Wealth 100, 200, 120 (millions): stock held	20 + 100 + 90	**210 (bonds also 210)**
Cash lent less borrowed	60 − 60	**0**
Market portfolio	210 ÷ 420 in each fund	**50% / 50%**

Watch out: Reading the indifference curve slope backwards: steeper means more risk averse.

Key takeaways

- A risk averse investor needs extra return for extra risk; utility = $E(R) - \frac{1}{2}A\sigma^2$, with variance in the penalty.
- The best portfolio is where the highest reachable indifference curve touches the frontier.
- Only the upper part of the minimum variance frontier, from the global minimum variance portfolio, is efficient.
- The lower the correlation, the more the frontier bows to the left.
- The capital allocation line has intercept Rf and slope equal to the Sharpe ratio; the tangency portfolio has the steepest line.
- With shared views and one risk-free rate, everyone holds the tangency portfolio, which is the market portfolio, and only the amount differs.

Quick check

1. Risk aversion 4, return 9%, risk 10%. What is the utility?
   Answer: $0.09 - 2 \times 0.01 = 0.07$
2. Rf 2%, tangency 10% return and 16% risk. Half in each: return and risk?
   Answer: 6% return, 8% risk
3. Shared views and one risk-free rate: which risky portfolio do all hold?
   Answer: The tangency portfolio = the market portfolio

Summary notes for the Charta whiteboard lesson. Examples use made-up figures. Watch the full lesson at charta-learning.com.