CHARTA LEARNING · LESSON NOTES

Probability rules and conditional expectations

CFA Level I · Quantitative Methods

You will learn: Combine chances with the addition and multiplication rules, find an expected value and variance, update a belief with Bayes' formula, measure covariance and correlation, and build a forecast from scenario averages.

Running example: Examples use made-up figures, such as a stock that falls 80% of the time in a recession (30% chance) and 10% of the time otherwise, and a bank test that flags fraud.

1. Probability and the addition rule

- A probability is a score from 0 (impossible) to 1 (certain). An event is something that might happen (A, B). P(A | B), read A given B, is the chance of A once B has happened, a different number from P(A) on its own.
- The addition rule gives the chance that A, or B, or both happen. Subtract the overlap, which is counted twice otherwise.
- Mutually exclusive events cannot both happen: the overlap is zero, so you simply add. The answer is never below either single chance or above 1.

P(A or B) = P(A) + P(B) − P(AB)

PICK ONE COMPANY: 40% PAY A DIVIDEND, 35% ARE LARGE, 15% ARE BOTH

P(A) + P(B)	0.40 + 0.35	**0.75**
Subtract the overlap	− 0.15	**0.60**
P(A or B)	dividend, large, or both	**0.60**

Watch out: Forgetting to subtract the overlap (giving 0.75 here).

2. Multiplication rule and independence

- The multiplication rule gives the joint probability that A and B both happen: first B happens, then, given B, A happens. Multiply the two steps; the joint chance is smaller than either part.
- Events are independent if knowing one happened tells you nothing about the other. Then P(A | B) = P(A) and P(AB) = P(A) × P(B). Two coin flips: 0.5 × 0.5 = 0.25 for two heads.
- Independent does not mean mutually exclusive. Exclusive events with real chances are dependent. Never assume independence unless the question says so.
- For a tree, multiply along each path, then add the paths that reach the outcome (the paths never overlap).

P(AB) = P(A | B) × P(B)

RECESSION 30%; STOCK FALLS 80% IN A RECESSION, 10% IF NOT

Recession and falls	0.30 × 0.80	**0.24**
No recession and falls	0.70 × 0.10	**0.07**
P(stock falls)	0.24 + 0.07	**0.31**
Independent?	0.80 is not 0.31	**No, dependent**

Watch out: Assuming independence when nobody told you so.

3. Expected value and variance

- A random variable is an uncertain quantity. The expected value is the probability-weighted average of all outcomes: the long-run average, not a promise. It may be a value that never occurs (a fair die averages 3.5). The probabilities must add to 1.
- Variance measures spread: take each outcome's distance from the expected value (not from zero), square it, weight by its probability, and add. Standard deviation is the square root, in the same units as the outcomes.

$E(X) = \Sigma\ p \times x$. Variance $= \Sigma\ p \times (x - E(X))^2$

RECESSION −10% (0.25), NORMAL 8% (0.50), BOOM 22% (0.25)

Weighted returns	−2.5 + 4.0 + 5.5	**7.0%**
Distances from 7%		**−17, +1, +15**
Squared		**289, 1, 225**
Variance	72.25 + 0.5 + 56.25	**129**
Standard deviation	√129	**11.36%**

Watch out: Measuring variance from zero, or without weighting by probability.

4. Bayes' formula

- Bayes' formula updates a belief with new evidence. The prior P(A) is the belief before the evidence. The likelihood P(B | A) is how probable the evidence is if A is true. P(B) is the total chance of the evidence. The posterior P(A | B) is the updated belief.
- Strong evidence moves a belief a long way, but the starting point always matters. A rare event stays fairly unlikely even after a positive test, because false alarms from the common case outnumber real catches.

$P(A \mid B) = P(B \mid A) \times P(A) \div P(B)$

FRAUD TEST: 1% OF ACCOUNTS ARE FRAUD; THE TEST CATCHES 90% OF FRAUD AND WRONGLY FLAGS 5% OF HONEST ACCOUNTS

Fraud and flagged	0.01 × 0.90	**0.0090**
Honest and flagged	0.99 × 0.05	**0.0495**
Any flag	0.0090 + 0.0495	**0.0585**
P(fraud | flag)	0.0090 ÷ 0.0585	**15.4%**
Counting 10,000 accounts	90 frauds flagged; 495 honest flagged	**false alarms > 5 to 1**

Watch out: Ignoring the prior.

5. Covariance and correlation

- Covariance measures whether two returns tend to rise and fall together: positive means they move together, negative means they offset. Its size depends on units, so you cannot compare it between pairs.
- Correlation divides covariance by the two standard deviations. It has no units and lies between −1 and +1.
- Limits: correlation only captures straight-line links (a curve can give zero); it says nothing about the size of moves; it never proves one thing causes another, since a third factor such as the economy may drive both.

$Cov(A,B) = \Sigma\ (a - \bar{a})(b - \bar{b}) \div (n - 1)$. $\rho = Cov(A,B) \div (\sigma A \times \sigma B)$

A: 2, 4, 6, 8%. B: 3, 2, 7, 8%. BOTH AVERAGES 5%

Distances for A; for B		**−3, −1, +1, +3; −2, −3, +2, +3**
Multiply the pairs		**6, 3, 2, 9**
Covariance	20 ÷ 3	**6.67**
Std dev of A; of B	√(20 ÷ 3); √(26 ÷ 3)	**2.58; 2.94**
Correlation	6.67 ÷ (2.58 × 2.94)	**0.88**

Watch out: Reading correlation as proof of cause, or as proof of no relationship when it is zero.

6. Conditional expectations

- A conditional expected value E(X | S) is the average outcome given that scenario S has happened; conditional variance is the spread inside that scenario, measured from that scenario's own average, not the overall average.
- The overall forecast weights each scenario's average by the chance of the scenario, and the chances must add to 1. When news changes the odds, only the weights move.

> **E(X) = Σ P(S) × E(X | S). Var(X | S) = Σ p × (x − E(X | S))²**

EPS: EXPANSION (0.6): $3.00 (0.7) OR $2.50 (0.3); RECESSION (0.4): $2.00 OR $1.40 (0.5 EACH)

E(EPS | expansion)	0.7 × 3.00 + 0.3 × 2.50	**$2.85**
Variance | expansion	SD about $0.23	**0.0525**
E(EPS | recession)	0.5 × 2.00 + 0.5 × 1.40	**$1.70**
Variance | recession	SD $0.30	**0.09**
Overall EPS	0.6 × 2.85 + 0.4 × 1.70	**$2.39**
After weak data (weights 0.4, 0.6)	0.4 × 2.85 + 0.6 × 1.70	**$2.16**

Watch out: Using the overall average instead of the scenario's own average in a conditional variance.

Key takeaways

- Addition rule: P(A or B) = P(A) + P(B) − P(AB). Multiplication rule: P(AB) = P(A | B) × P(B).
- Independent is not the same as mutually exclusive; do not assume independence.
- Expected value is a probability-weighted average; variance is the weighted squared distance from it.
- Bayes: posterior = likelihood × prior ÷ total chance of the evidence; do not ignore the prior.
- Correlation = covariance ÷ (σA × σB), between −1 and +1; it is not proof of cause.
- Overall forecast = sum of scenario chance × scenario average.

Quick check

1. P(A) = 0.30, P(B) = 0.50, P(AB) = 0.10. What is P(A or B)?
   Answer: 0.30 + 0.50 − 0.10 = 0.70.
2. EPS is $5 if the economy grows (chance 0.6) and $3 otherwise. Expected EPS?
   Answer: 0.6 × 5 + 0.4 × 3 = $4.20.
3. Cov(A,B) = 12, σA = 4, σB = 5. What is the correlation?
   Answer: 12 ÷ (4 × 5) = 0.6.

Summary notes for the Charta whiteboard lesson. Examples use made-up figures. Watch the full lesson at charta-learning.com.