CHARTA LEARNING · LESSON NOTES

The time value of money, part 1

CFA Level I · Quantitative Methods

You will learn: How to move a single amount forward and backward in time, value annuities and perpetuities, and turn a stated rate into an effective annual rate so that quoted rates can be compared fairly.

Running example: Examples use made-up figures in pounds, such as £1,000 invested for three years at 8% a year, and £10,000 promised in six years when safe accounts pay 5%.

1. Future value and compounding

- Money has a date attached, and the date changes its value. The amount today is the present value (PV); the amount at a later date is the future value (FV).
- Compounding means each year's interest joins the balance and then earns interest itself, so growth is a curve, not a straight line.
- r is the interest rate for one period (the gap between interest payments, so it may be a year or a month). n is the number of periods you wait.
- Golden rule: the rate and the number of periods must describe the same length of time. Ask what one period is, then make the rate and the count match it.

$FV = PV \times (1 + r)^n$

£1,000 FOR 3 YEARS AT 8% A YEAR, INTEREST ADDED YEARLY

Growth factor	1.08^3	**1.2597**
Future value	1,000 × 1.2597	**£1,259.71**
Simple interest would give	no interest on interest	**£1,240.00**
Interest on interest	1,259.71 − 1,240	**£19.71**

Watch out: Using 6% with 20 periods, or 1.5% with 5 periods, when interest is quarterly. Convert first, calculate second.

2. Quarterly interest and present value

- Quoted 6% a year added every quarter means a rate per quarter of 6% ÷ 4 = 1.5%, and 5 years × 4 = 20 quarters.
- Present value runs the formula backwards. Discounting is moving money back in time, and r is now the discount rate.
- A higher discount rate gives a smaller PV, and a longer wait gives a smaller PV. The discount rate is the return on the best alternative of similar risk that you give up.
- Long-dated payments are very sensitive to the rate. Two analysts can do correct arithmetic and still disagree on value because they choose different rates.

$PV = FV \div (1 + r)^n$

£5,000 FOR 5 YEARS AT 6% A YEAR, ADDED QUARTERLY

Rate per quarter, n	6% ÷ 4 = 1.5%; 5 × 4	**0.015; 20**
Growth factor	1.015^{20}	**1.3469**
Yearly 6% for comparison	1.06^5	**1.3382**
Future value	5,000 × 1.3469	**£6,734.28**

Watch out: Forgetting that the extra over yearly interest comes only from interest being added more often.

3. Present value of a promise

- To value a promise, discount it at the return you could earn on a safe alternative. Check the answer by growing it forward: it must return the promised amount.
- The same promise has a different present value at each discount rate, even though the promise has not changed.

PV = FV ÷ (1 + r)ⁿ

£10,000 PROMISED IN 6 YEARS; SAFE ACCOUNTS PAY 5% A YEAR		
Discount factor	1.05^6	**1.3401**
Present value	10,000 ÷ 1.3401	**£7,462.15**
Check: grow it back	7,462.15 × 1.3401	**£10,000**
At 3% / 8% / 10%	same promise, only the rate changes	**£8,374.84 / £6,301.70 / £5,644.74**

4. Perpetuities and annuities

- An annuity is equal payments (PMT), a fixed number of times, with the same gap between each. A perpetuity is equal payments for ever.
- Ordinary annuity: payments at the end of each period, so the first is one period away. Annuity due: payments at the start, so the first is today.
- A perpetuity is worth the payment divided by the rate: the sum that earns the payment as interest without being used up. Its formula assumes the first payment is one period away; if it starts today, multiply by (1 + r).
- An annuity is one perpetuity minus the tail you do not receive. The annuity formula is for an ordinary annuity; for an annuity due, multiply the answer by (1 + r) once.

Perpetuity: PV = PMT ÷ r. Ordinary annuity: PV = PMT × [1 − (1 + r)⁻ⁿ] ÷ r

ANNUITY OF £2,000 AT THE END OF EACH YEAR FOR 5 YEARS AT 6%		
Discount factor	1.06^{-5}	**0.7473**
1 − 0.7473, ÷ 0.06	0.2527 ÷ 0.06	**4.2124**
Present value	2,000 × 4.2124	**£8,424.73**
Annuity due	8,424.73 × 1.06	**£8,930.21**
Perpetuity of £2 at 5%	2 ÷ 0.05; first payment today: 40 × 1.05	**£40; £42**
£1,000 for 10 years at 5%	20,000 − (20,000 ÷ 1.05^{10} = 12,278.27)	**£7,721.73**

Watch out: Forgetting the annuity due step, applying it to an ordinary annuity by mistake, or using a wrong first-payment date. Read when the first payment arrives.

5. Stated rate and effective annual rate

- The stated (nominal) annual rate is the headline number and does not say how often interest is added. The effective annual rate (EAR) is what you really earn over one year once compounding is included.
- m is the number of compounding periods in a year: 4 for quarterly, 12 for monthly.
- More frequent compounding always earns more, but with a ceiling: continuous compounding gives e^stated − 1, where e is about 2.7183. Most of the gain comes in the first few steps.
- Never compare quoted rates directly. Convert both to EARs first.

EAR = (1 + stated ÷ m)ᵐ − 1

STATED RATE 8%, AND WHICH LOAN IS CHEAPER		
Quarterly (m = 4)	$1.02^4 - 1$	**8.24%**
Yearly / half-yearly / monthly	m = 1, 2, 12	**8.00% / 8.16% / 8.30%**
Continuous	e^0.08 − 1	**8.33%**
Lender A: 12%, added yearly		**EAR 12.00%**
Lender B: 11.8%, added monthly		**EAR 12.46%**
Cheaper loan	lower effective rate	**Lender A**

Watch out: Comparing stated rates, not EARs.

6. Mistakes to avoid

- Rate and periods on different bases: if interest is quarterly, use the quarterly rate and the number of quarters.
- Forgetting that the annuity formula assumes payments at the end; for payments at the start multiply by (1 + r).
- Using the perpetuity formula with a wrong first-payment date.
- Comparing stated rates directly. Before every question, draw a quick timeline, mark when each payment happens, then choose the formula.

Key takeaways

- $FV = PV \times (1 + r)^n$ grows money forward; $PV = FV \div (1 + r)^n$ discounts it back.
- Rate and periods must be on the same footing before you use the formula.
- A higher discount rate or a longer wait gives a smaller present value.
- Perpetuity = PMT ÷ r; ordinary annuity = $PMT \times [1 - (1 + r)^{-n}] \div r$; annuity due = ordinary × (1 + r).
- $EAR = (1 + \text{stated} \div m)^m - 1$, and more frequent compounding has a ceiling.
- Convert quoted rates to EARs before comparing them.

Quick check

1. £200 is invested for 2 years at 10% a year. What is the future value?
   Answer: $200 \times 1.1^2 = £242$.
2. A perpetuity pays £300 a year; the rate is 6%. What is its present value?
   Answer: 300 ÷ 0.06 = £5,000.
3. 12% stated, added monthly. Is the EAR above or below 12%?
   Answer: Above: $1.01^{12} - 1 = 12.68\%$.